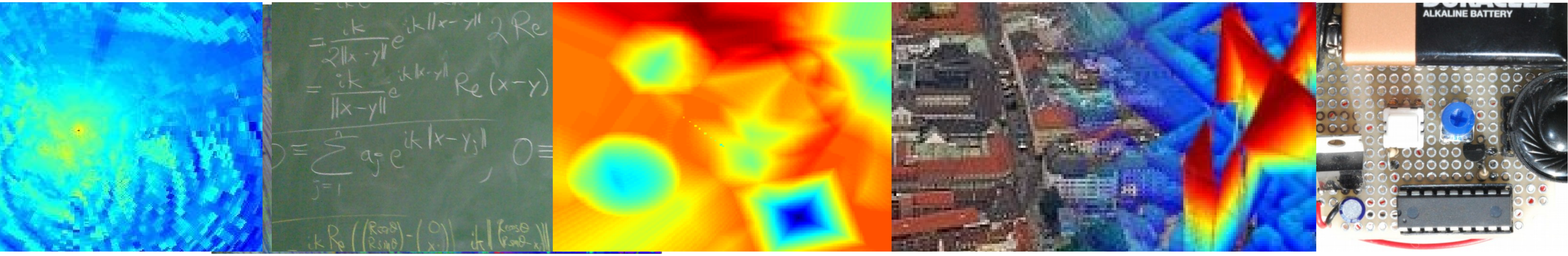
Sheaves and Numerical Analysis



Michael Robinson

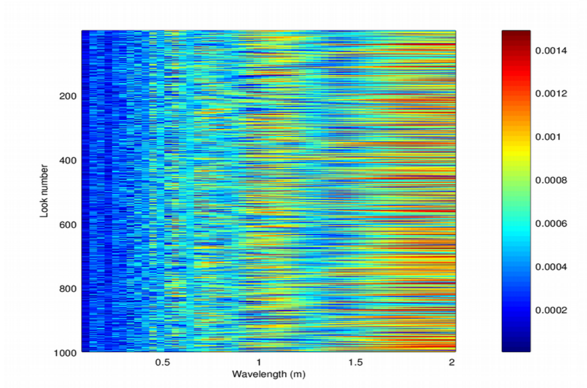
Acknowledgements

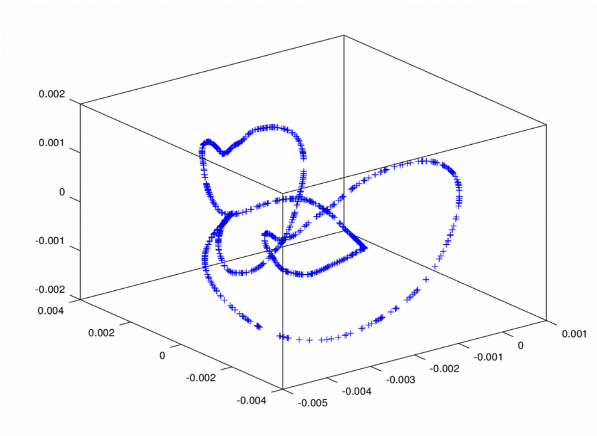
- Collaborators:
  - Cliff Joslyn, Katy Nowak, Brenda Praggastis, Emilie Purvine (PNNL)
  - Chris Capraro, Grant Clarke, Janelle Henrich (SRC)
- Students:
  - Olivia Chen
  - Philip Coyle
  - Samara Fantie
  - Jackson Williams
- Funding: Reza Ghanadan, John Paschkewitz (DARPA), Kyle Becker (ONR)

http://www.drmichaelrobinson.net/

Complex behaviors of dynamical systems

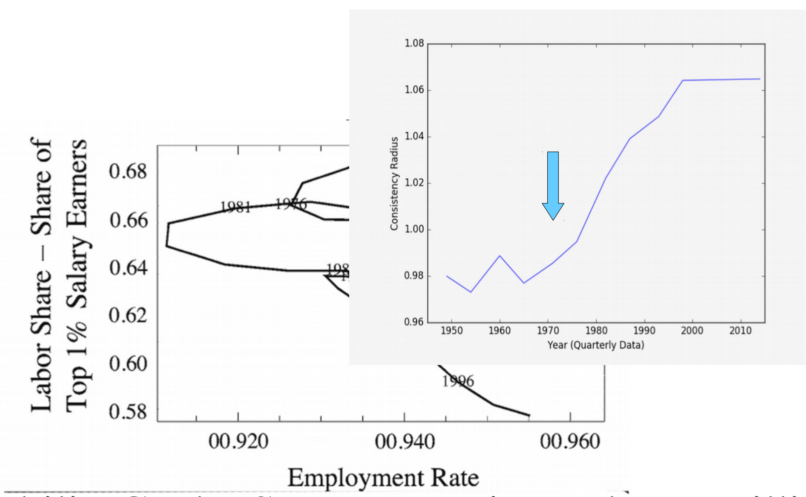
Sonar echos





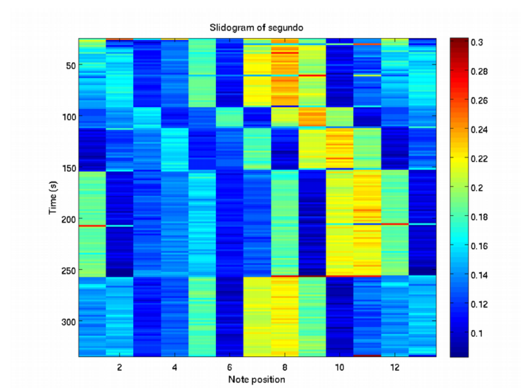
Jason Summers, Brian DiZio, Sean Fennell, Jen Dumiak

Econometrics



Philip Coyle, Samara Fantie, Jackson Williams

Improvisational Music



Fernando Benadon, Andy McGraw

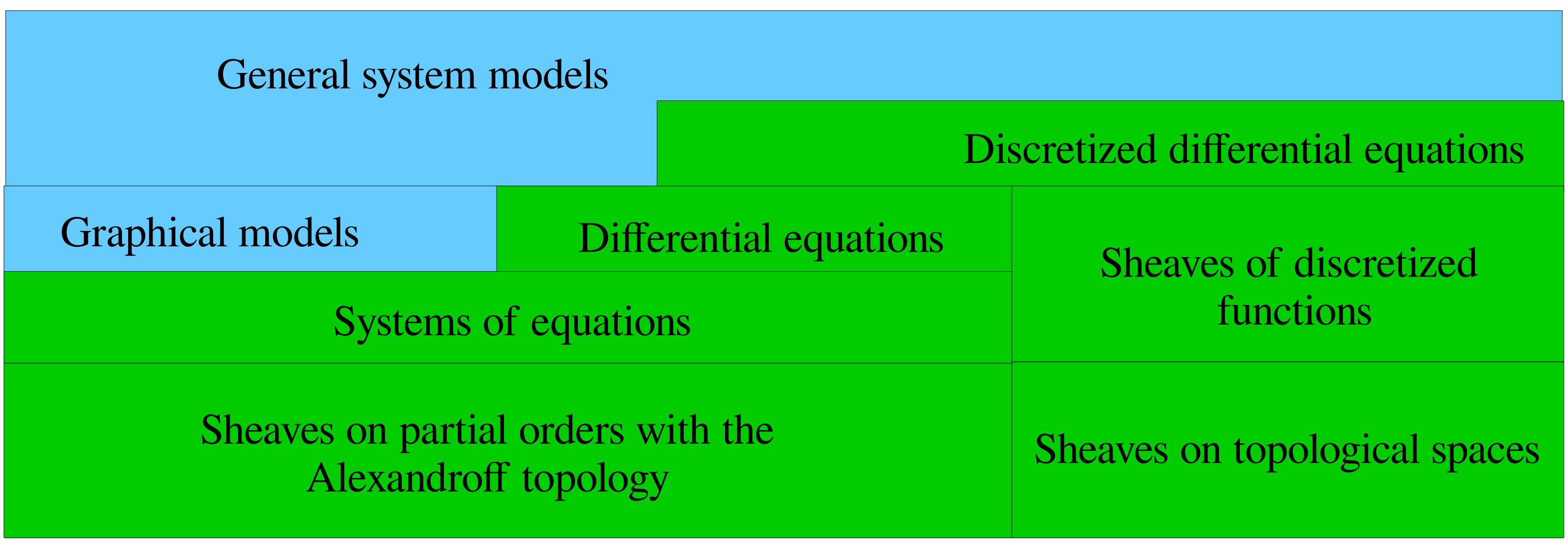
Key points

Question: Does discretization destroy dynamical structure?

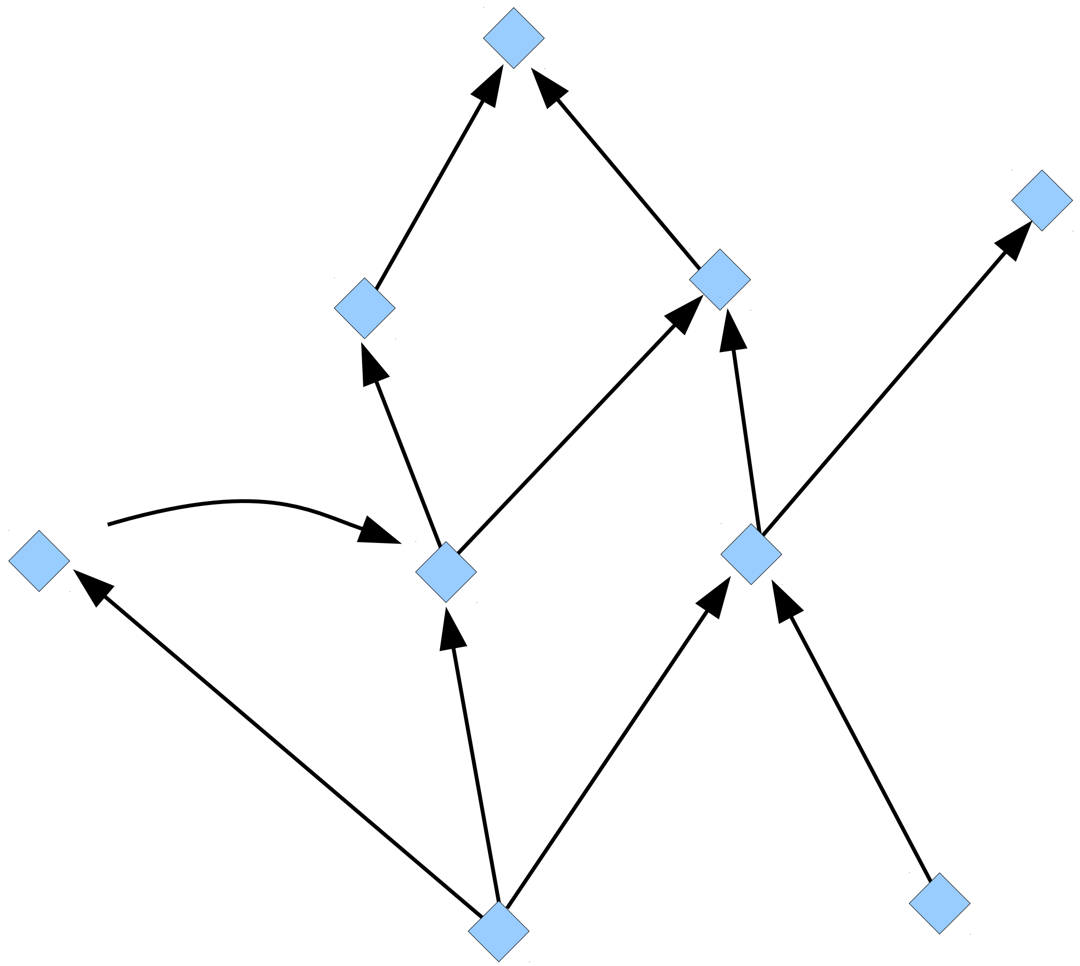
- Differential equations can be **encoded** as *sheaves*
- *Consistency* of numerical methods is characterized by the **commutativity** of *sheaf morphisms*
- Time evolution induces a **universal** sheaf morphism

The big picture

- *Partial orders* describe the relationships between variables in a system… order relations correspond to (differential) operators
- Every partial order has a natural topology, the *Alexandroff topology*
  - *Presheaves* and *sheaves* “are the same thing” in this topology, since the gluing axiom is satisfied trivially
  - Commutativity is the only actual constraint on a sheaf diagrams

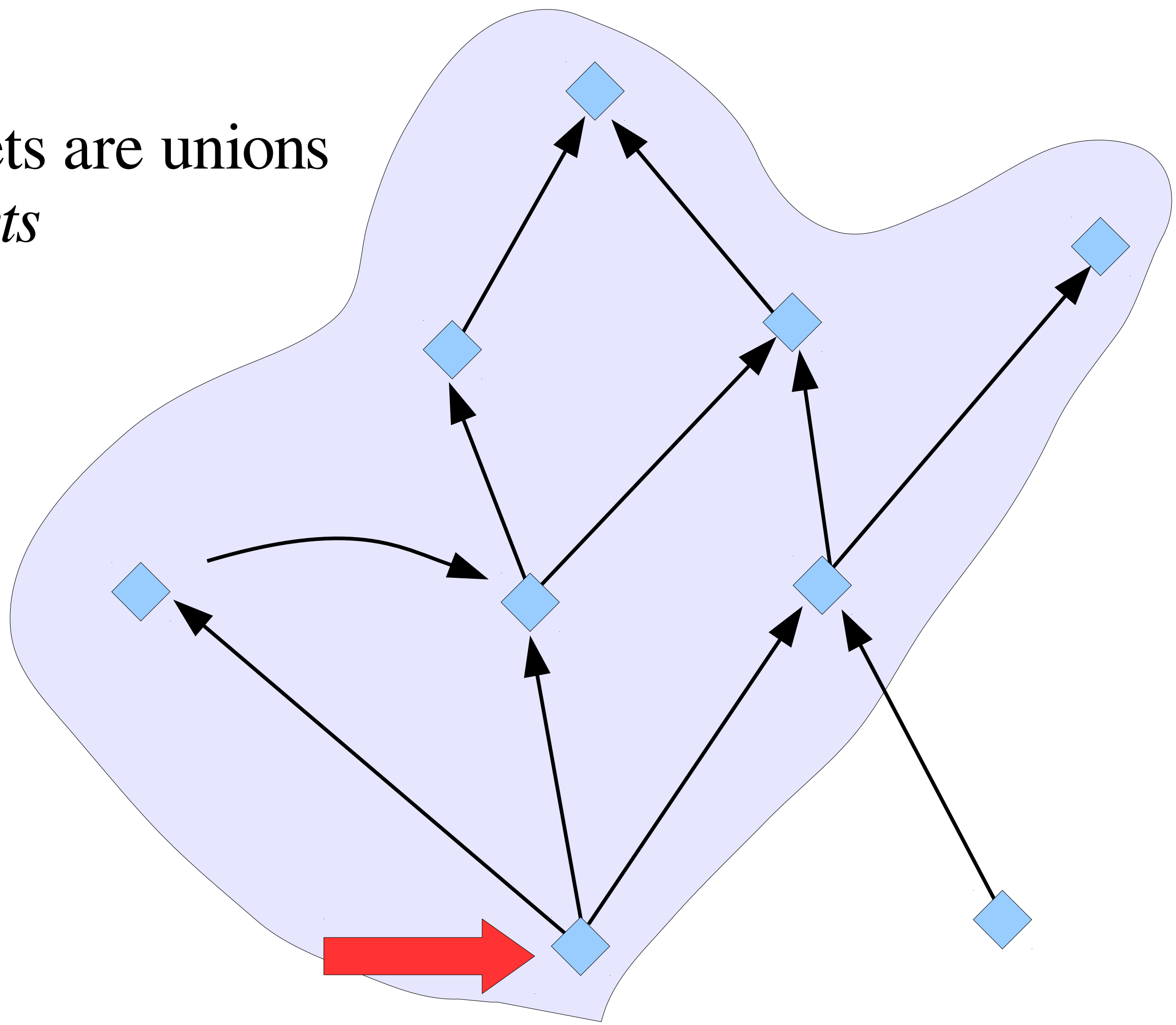


Topologizing a partial order



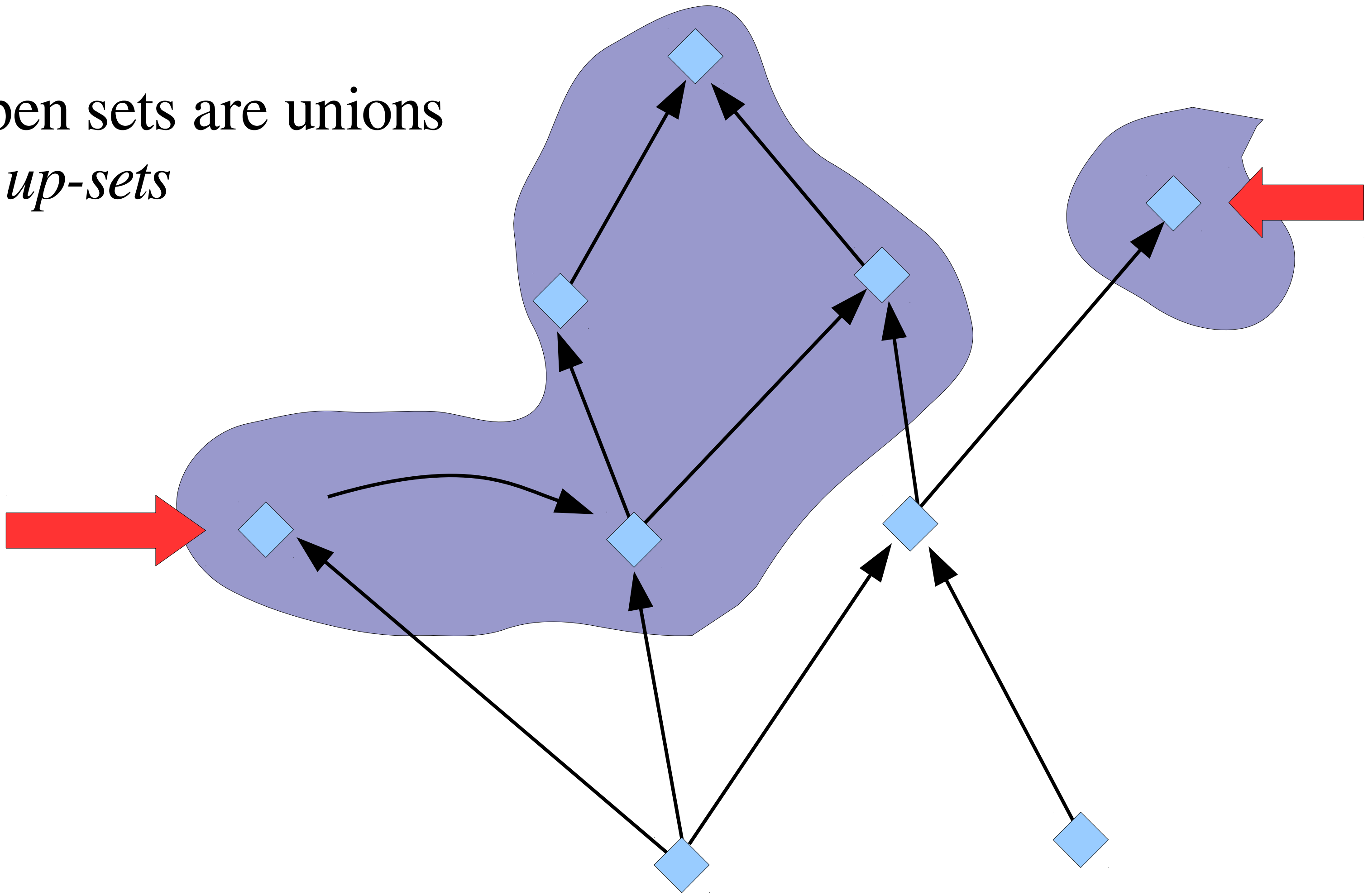
Topologizing a partial order

Open sets are unions of *up-sets*



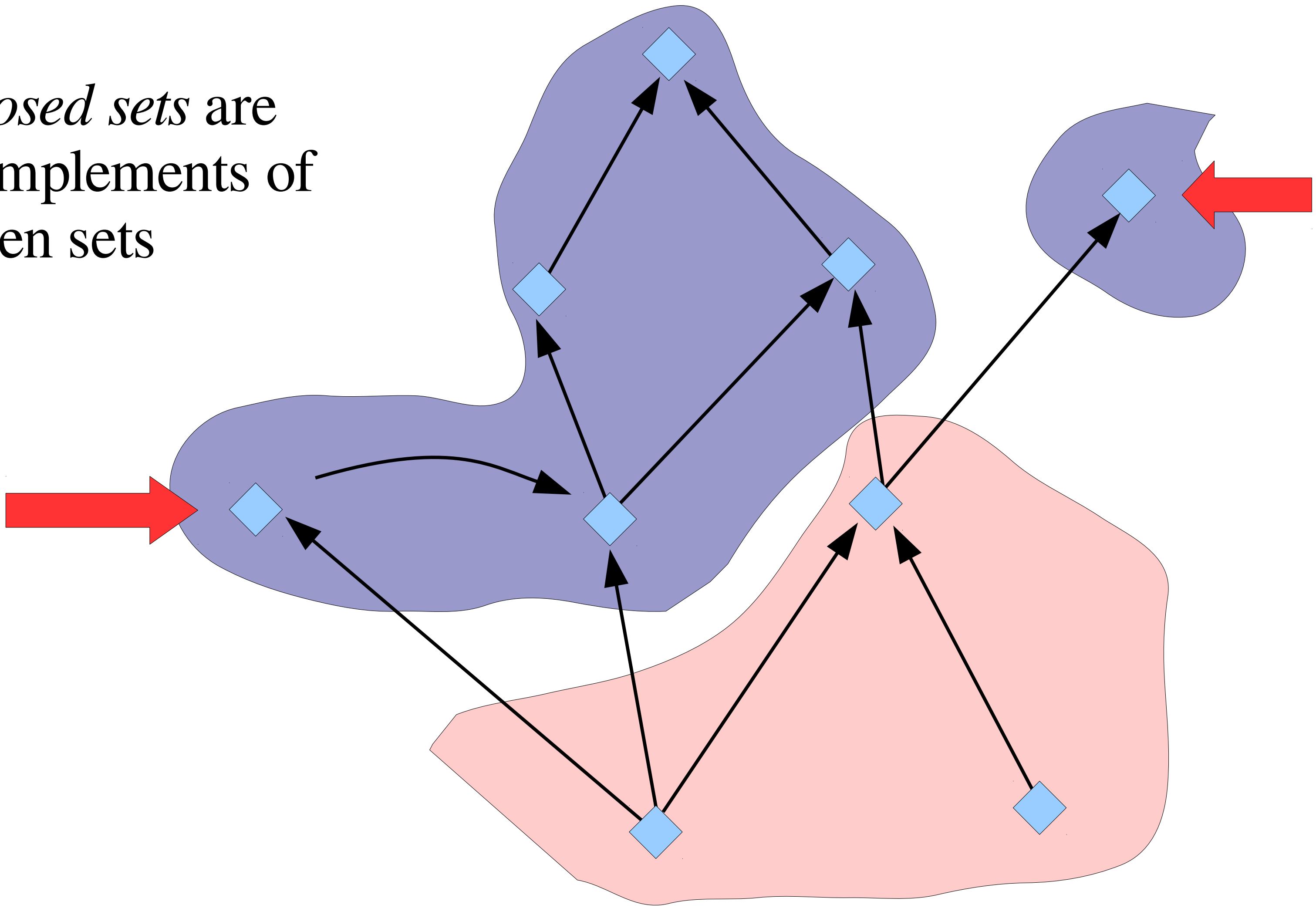
Topologizing a partial order

Open sets are unions of *up-sets*



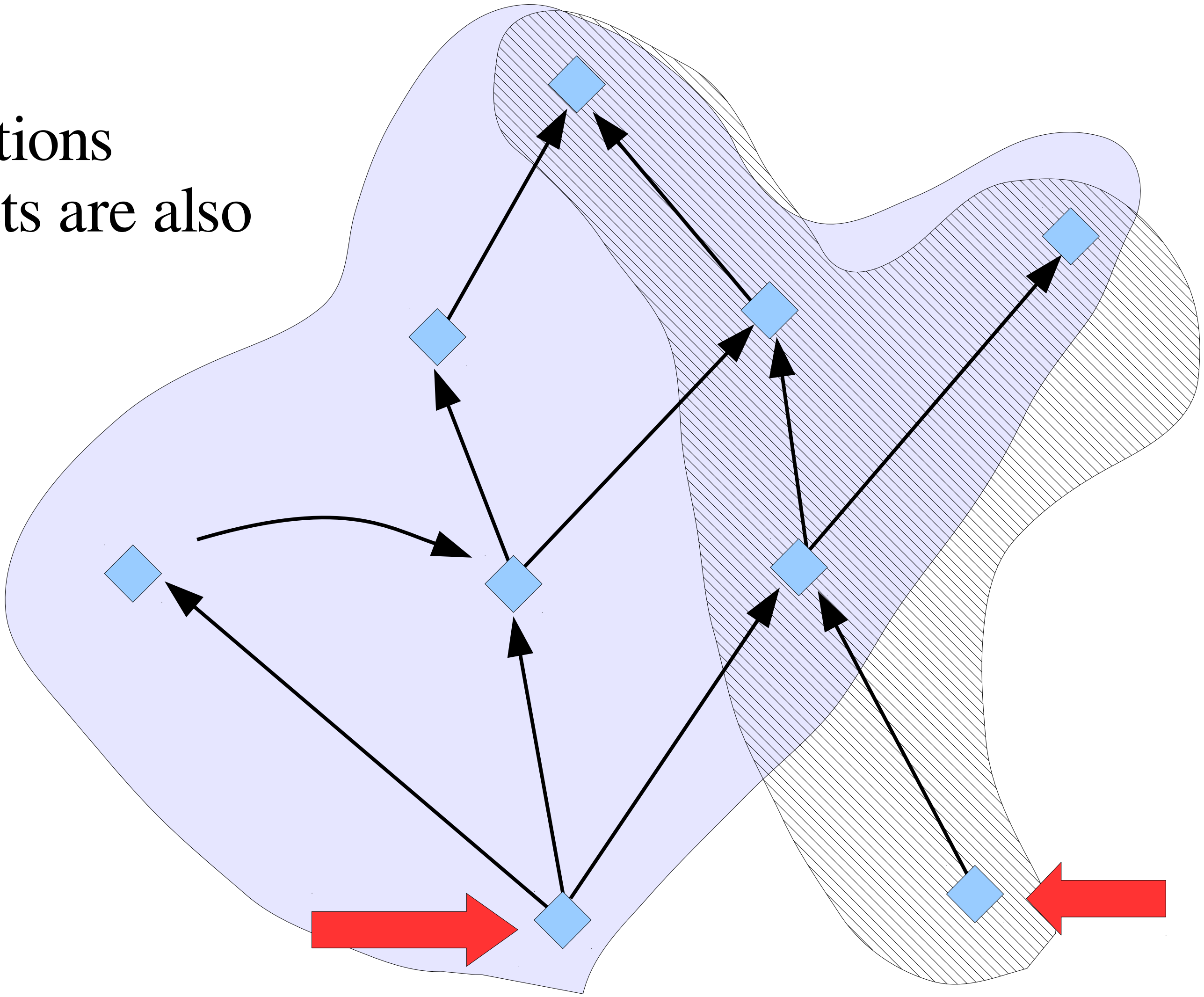
Topologizing a partial order

Closed sets are complements of open sets



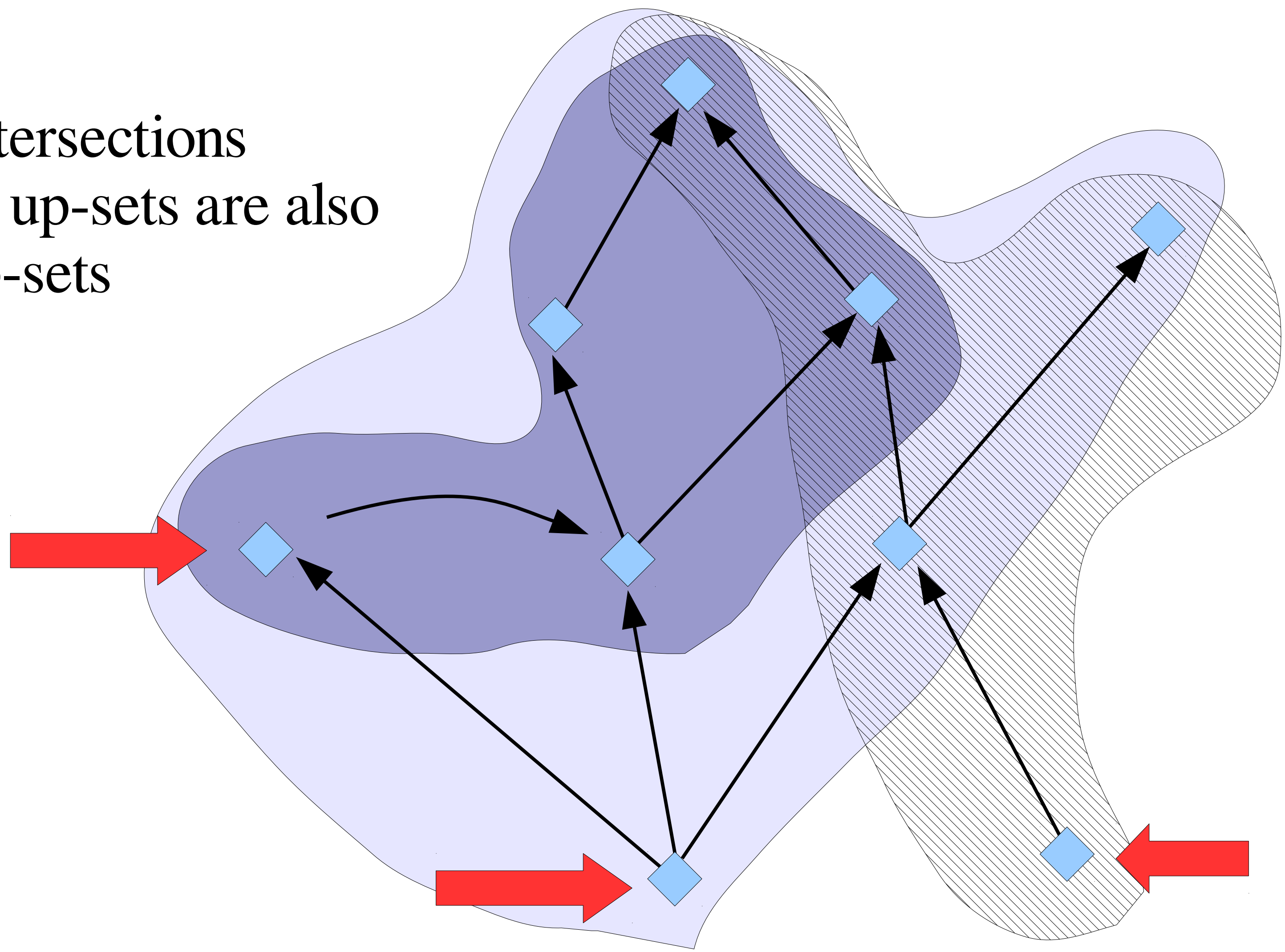
Topologizing a partial order

Intersections of up-sets are also up-sets



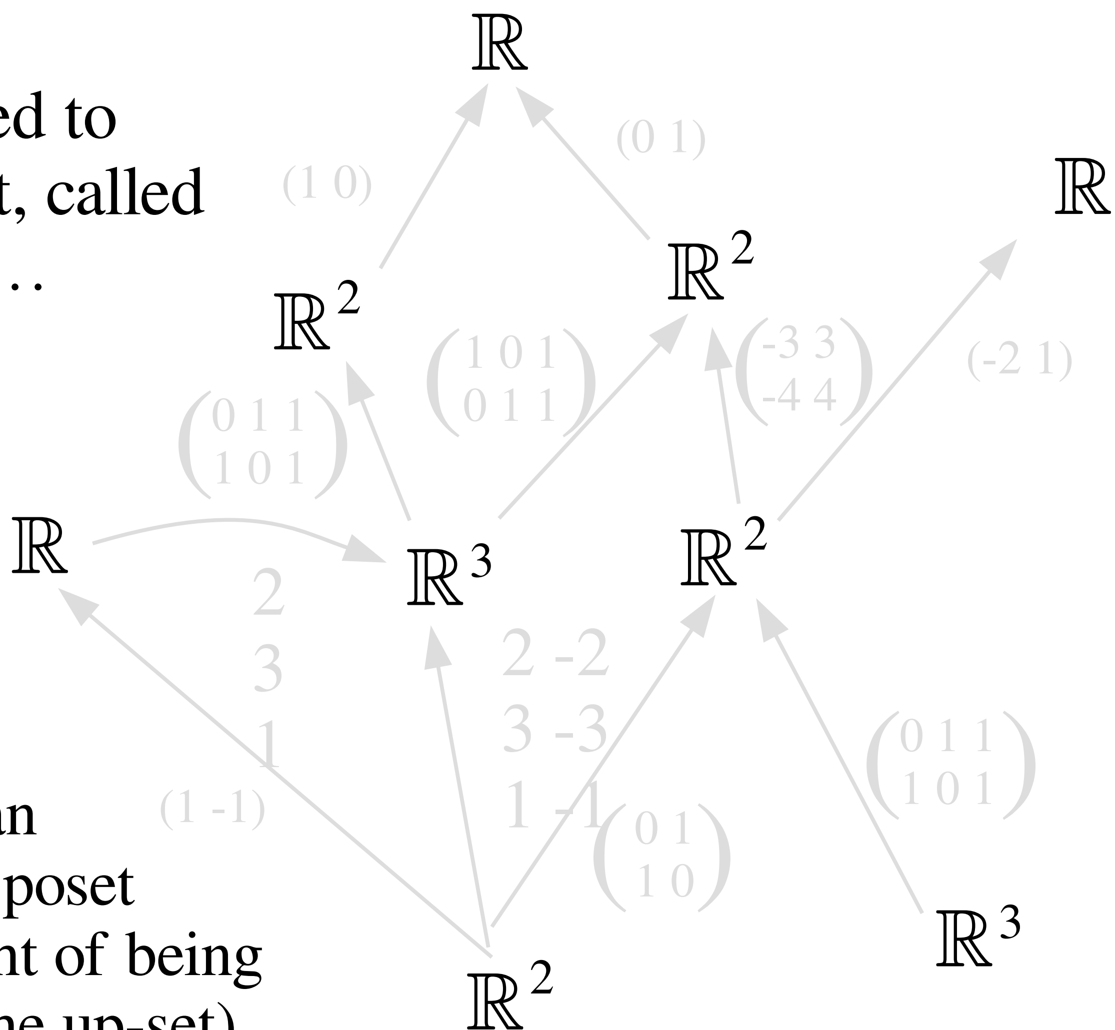
Topologizing a partial order

Intersections
of up-sets are also
up-sets



A *sheaf* on a poset is...

A set assigned to each element, called a *stalk*, and …

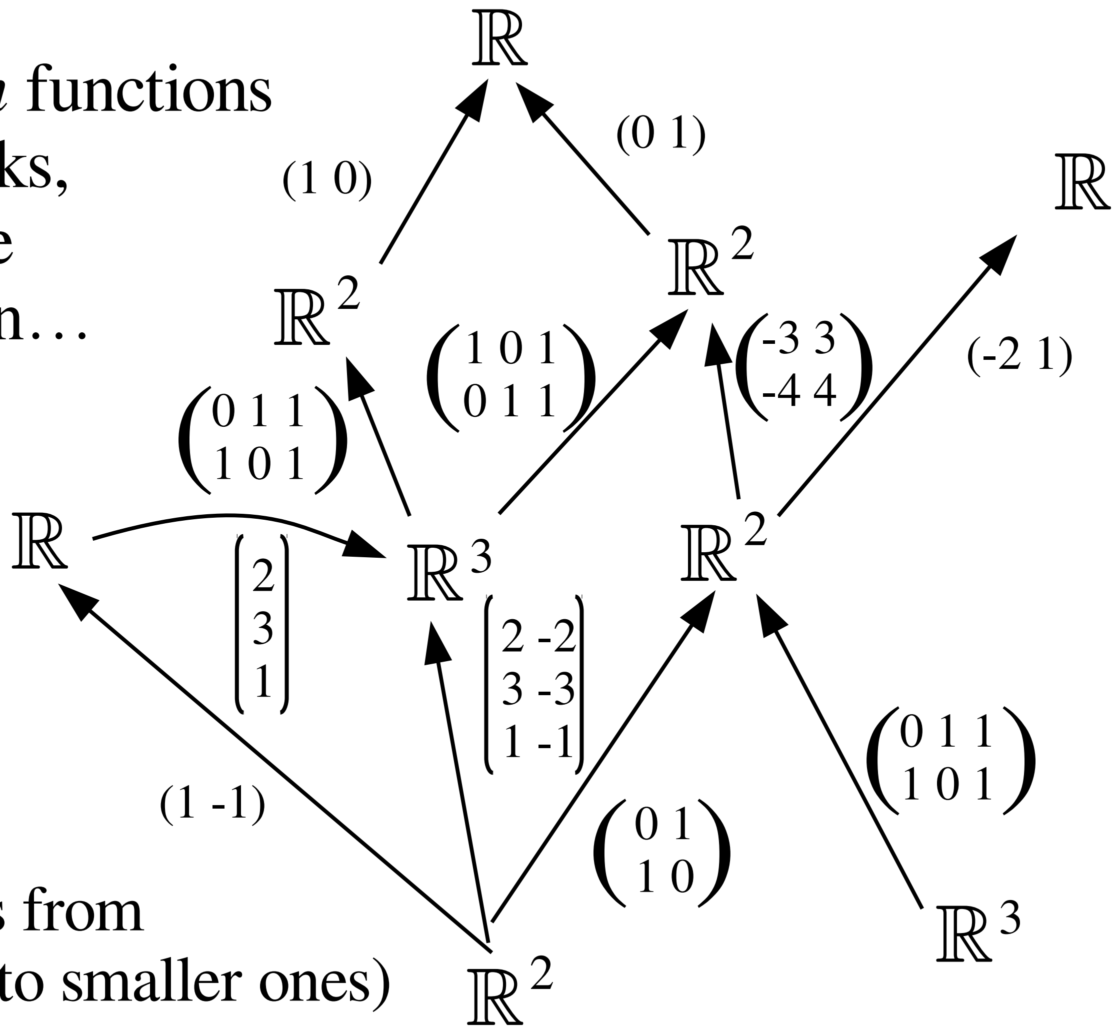


(The stalk on an element in the poset is better thought of being associated to the up-set)

This is a *sheaf* **of** vector spaces **on** a partial order

A *sheaf* on a poset is...

… *restriction* functions between stalks, following the order relation…

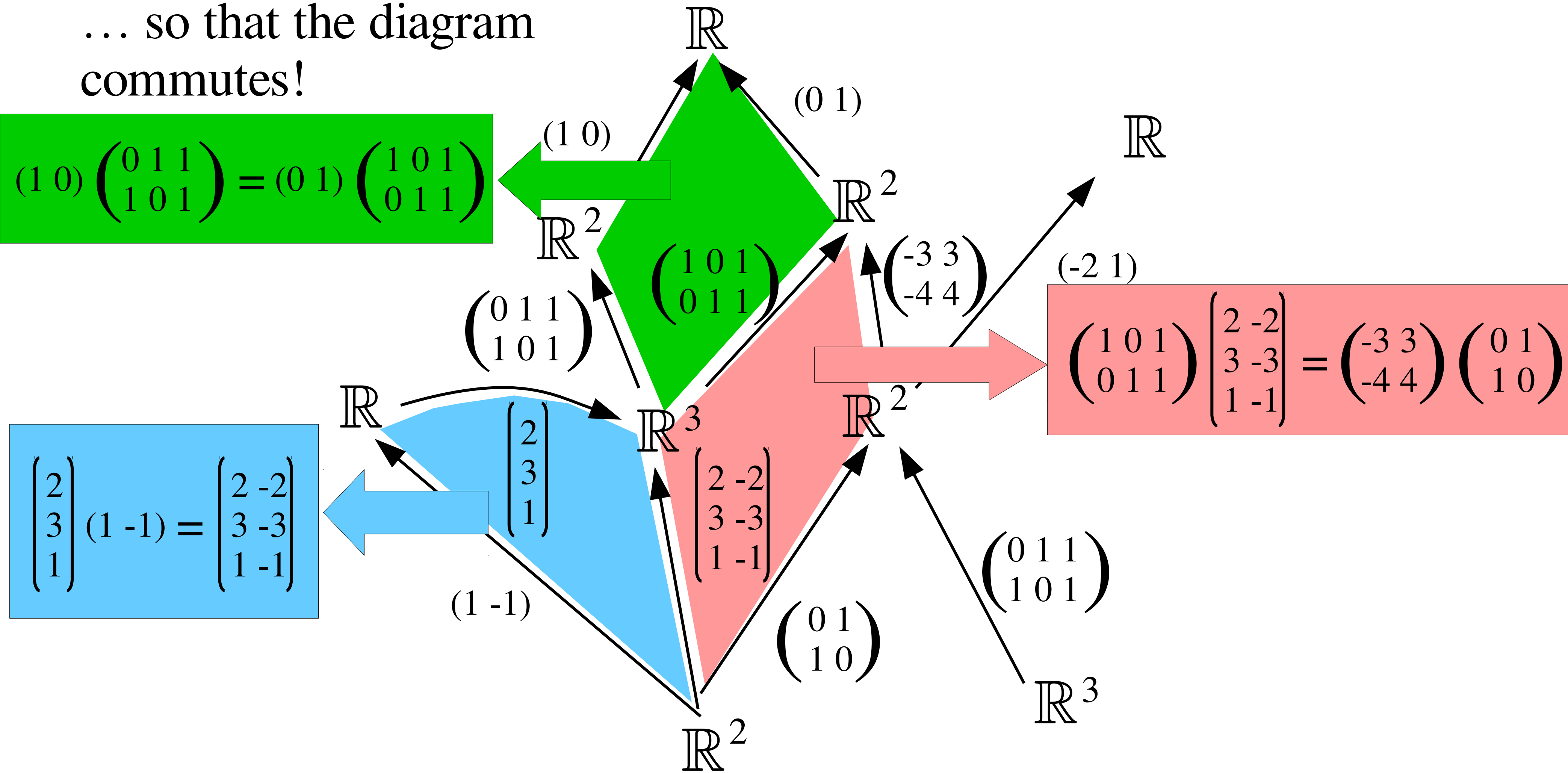


(“Restriction” because it goes from bigger up-sets to smaller ones)

This is a *sheaf* **of** vector spaces **on** a partial order

A *sheaf* on a poset is...

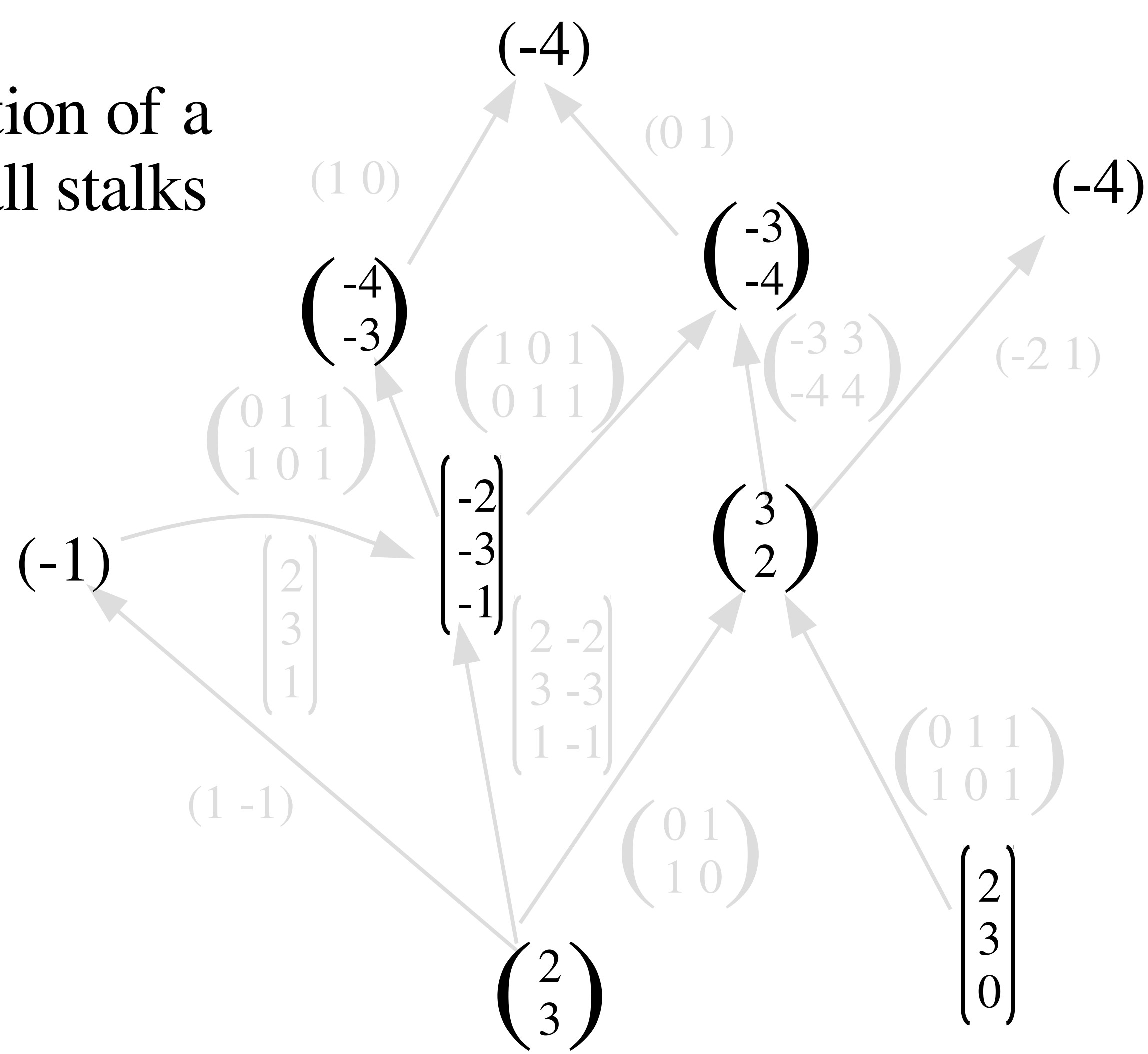
… so that the diagram commutes!



This is a *sheaf* **of** vector spaces **on** a partial order

An *assignment* is...

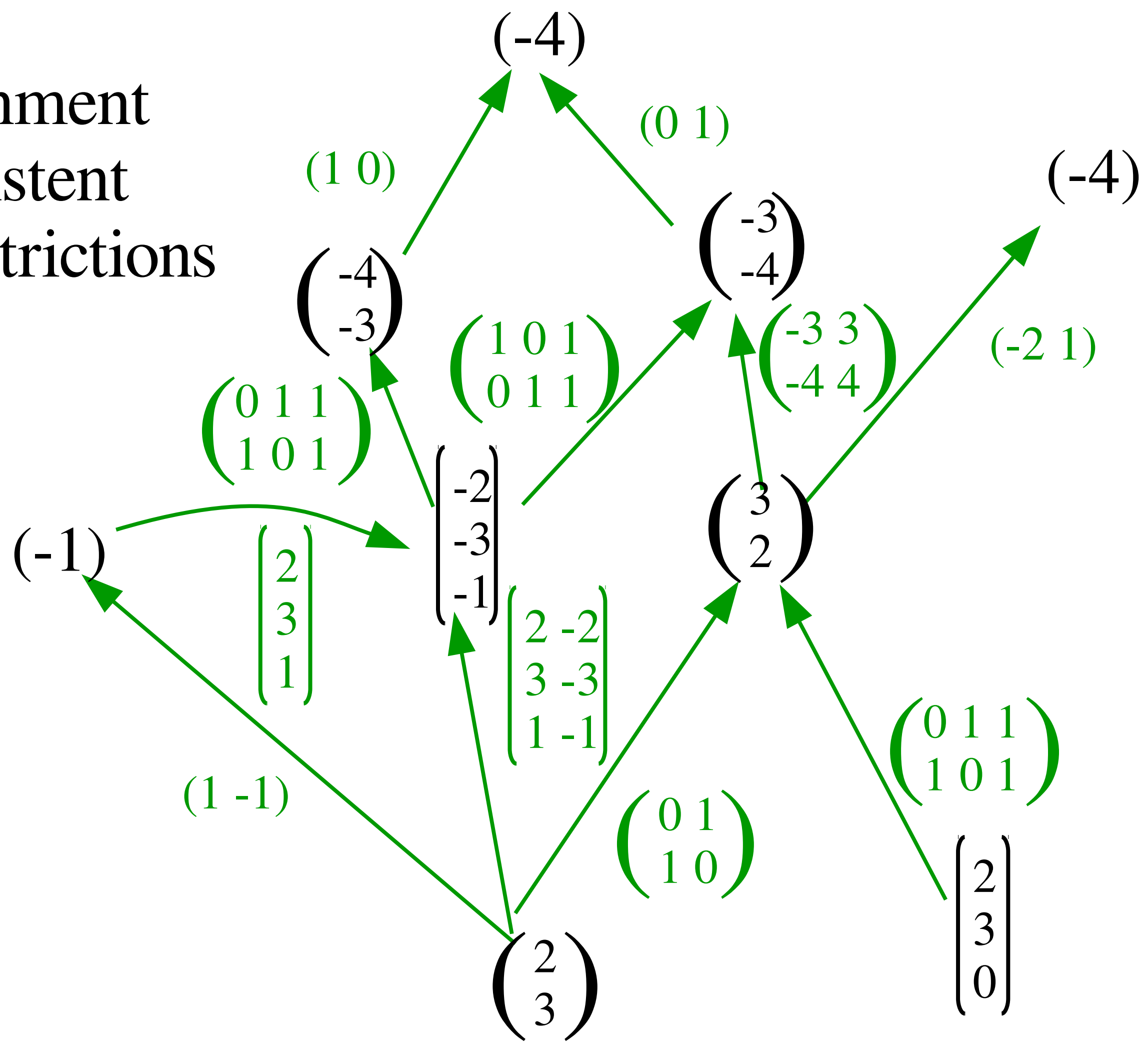
… the selection of a value from all stalks



The term *serration* is more common, but perhaps more opaque.

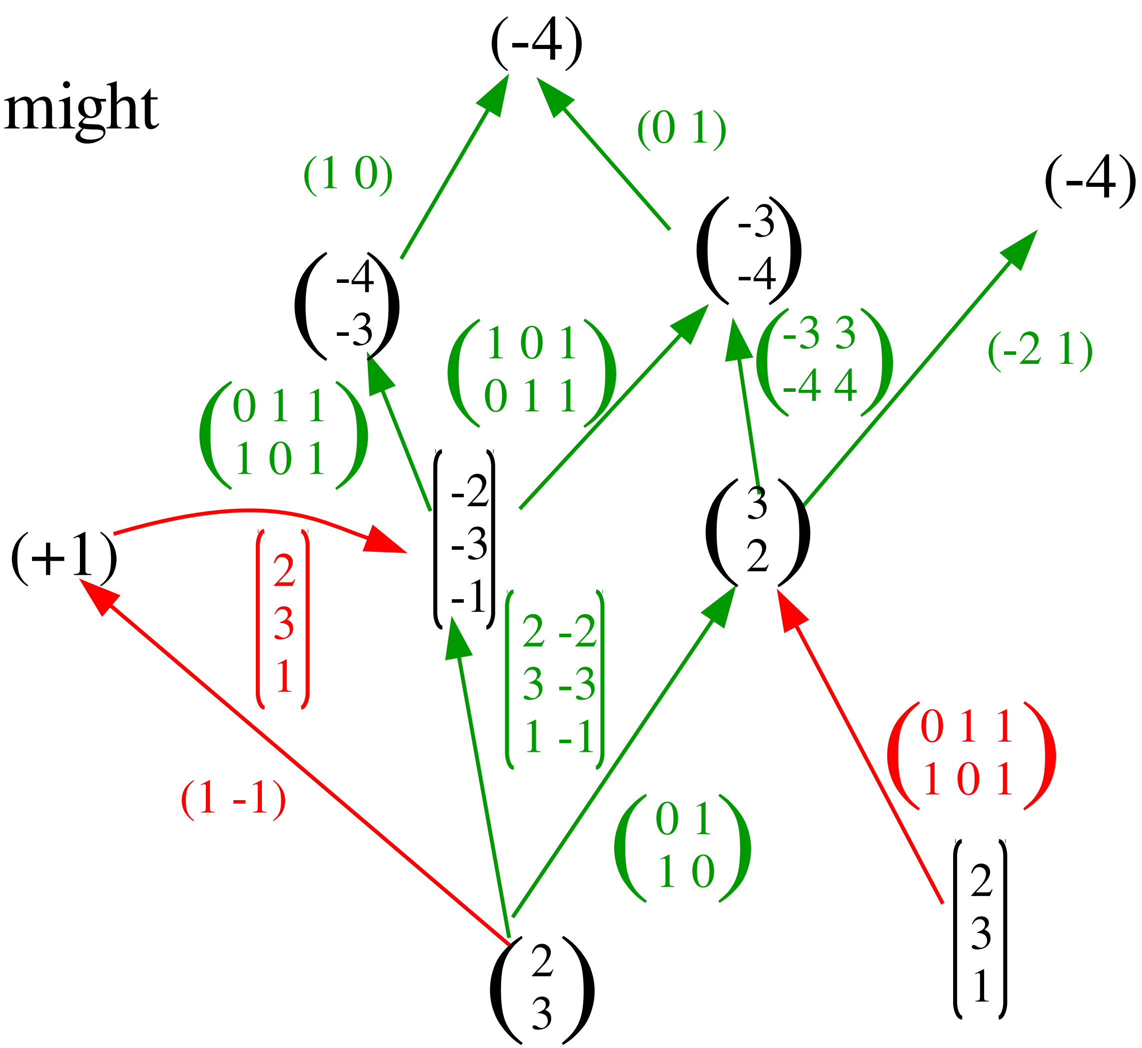
A *global section* is...

… an assignment that is consistent with the restrictions



Some assignments aren't consistent

… but they might be partially consistent



Goodwin macroeconomic model

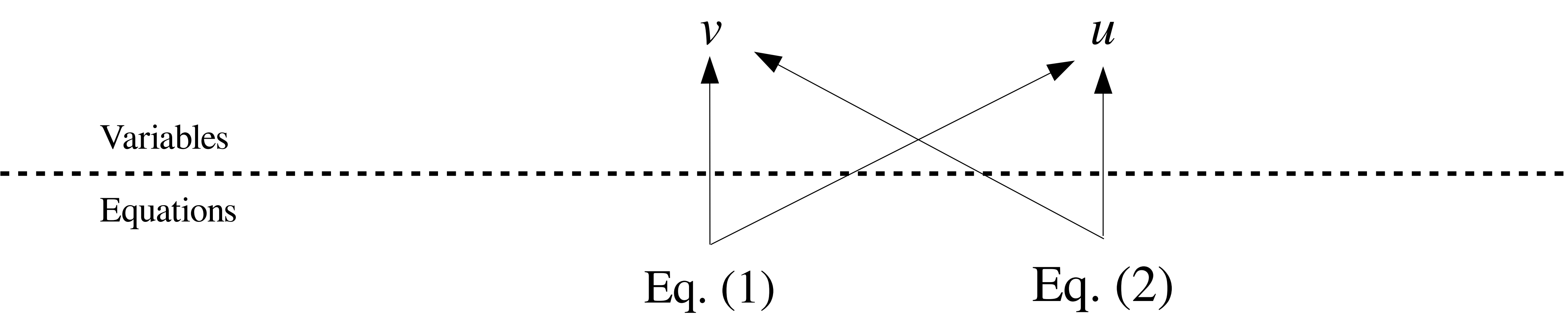
- A simple description of a national economy:

$$\dot{v} = \quad v(t)\left(\frac{1}{\sigma} - (\alpha+\beta) - \frac{u(t)}{\sigma}\right) \qquad (1)$$

v = Employment rate

$$\dot{u} = \quad u(t)\left(-(\alpha+\gamma) + (\rho v(t))\right). \qquad (2)$$

u = Workers' share of income



Goodwin macroeconomic model

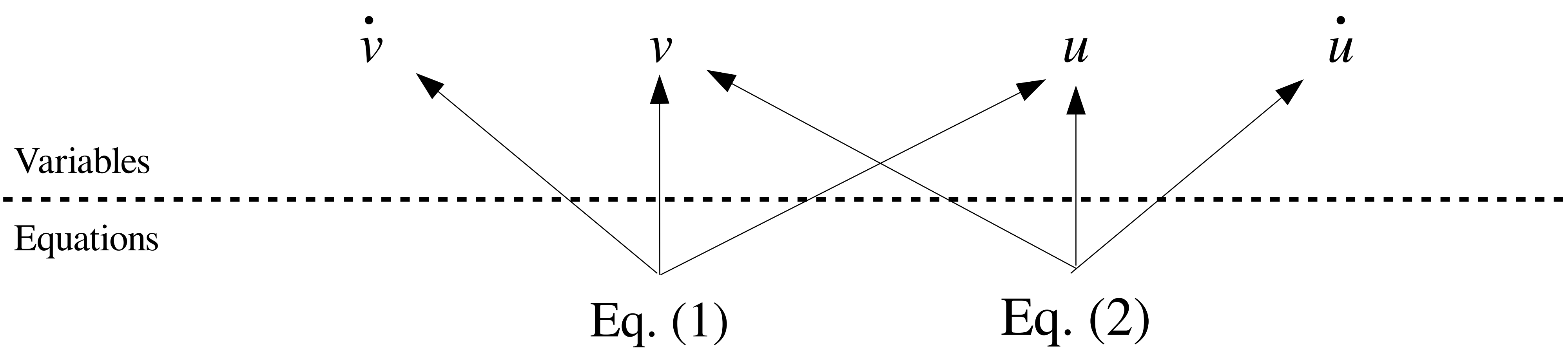
- A simple description of a national economy:

$$\dot{v} = \quad v(t)\left(\frac{1}{\sigma} - (\alpha+\beta) - \frac{u(t)}{\sigma}\right) \qquad (1)$$

v = Employment rate

$$\dot{u} = \quad u(t)\left(-(\alpha+\gamma) + (\rho v(t))\right). \qquad (2)$$

u = Workers' share of income



Goodwin macroeconomic model

- A simple description of a national economy:

$$\dot{v} = v(t)\left(\frac{1}{\sigma} - (\alpha + \beta) - \frac{u(t)}{\sigma}\right) \quad (1)$$

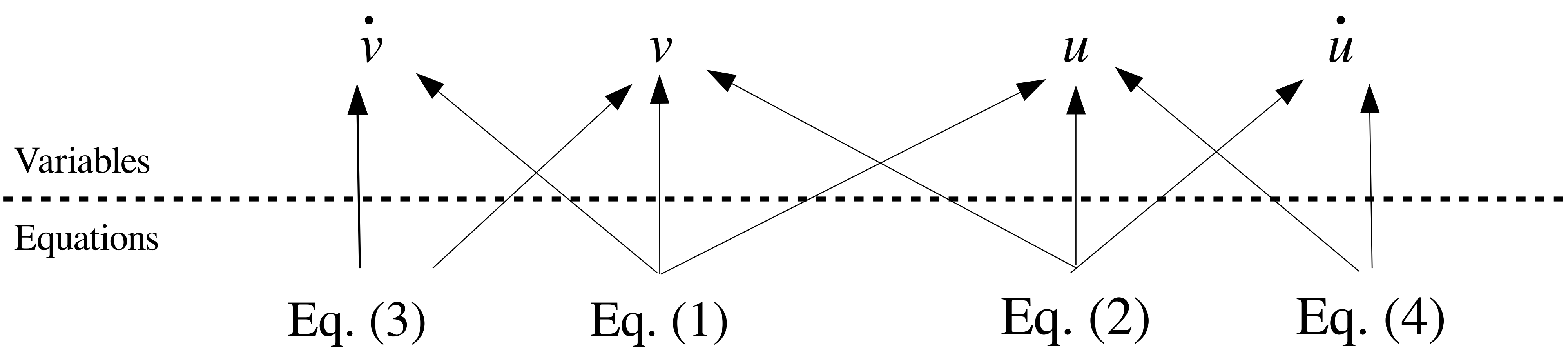
v = Employment rate

$$\dot{u} = u(t)\left(-(\alpha + \gamma) + (\rho v(t))\right). \quad (2)$$

u = Workers' share of income

$$\dot{v} = dv/dt \quad (3)$$

$$\dot{u} = du/dt \quad (4)$$



Goodwin macroeconomic model

- A simple description of a national economy:

$$\dot{v} = \quad v(t)\left(\frac{1}{\sigma} - (\alpha+\beta) - \frac{u(t)}{\sigma}\right) \qquad (1)$$

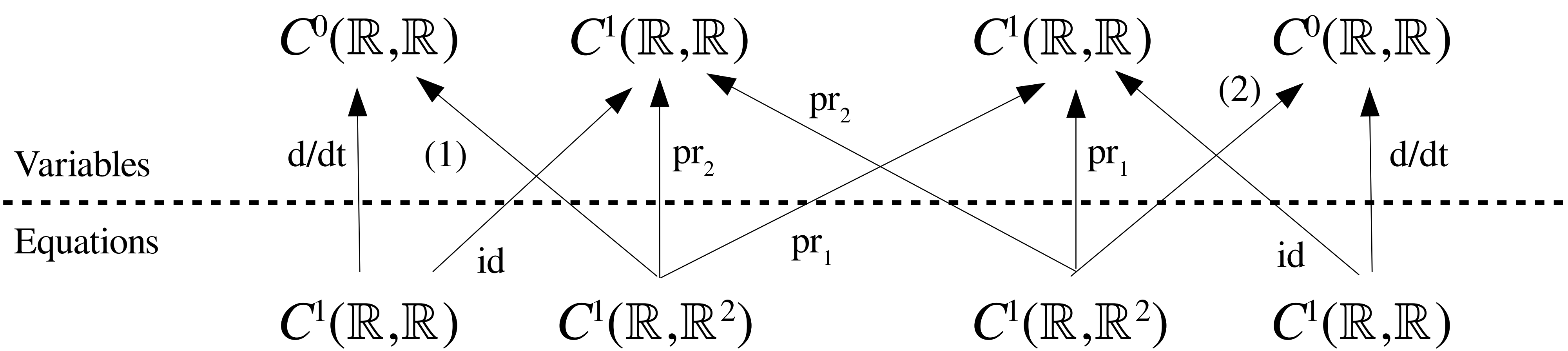
v = Employment rate

$$\dot{u} = \quad u(t)\left(-(\alpha+\gamma) + (\rho v(t))\right). \qquad (2)$$

u = Workers' share of income

$$\dot{v} = dv/dt \qquad (3)$$

$$\dot{u} = du/dt \qquad (4)$$



Multi-equation sheaves

- Theorem: (R.) For every system of equations, there is a sheaf whose global sections are solutions
  - Base poset has two levels: Equations < Variables
  - Stalk over each variable is that variable's set of possible values
  - Stalk over an equation is a subset of the product of the variables involved
  - Restriction maps are projections

  Proof: Straightforward once the construction is built!

Source: M. Robinson, "Sheaf and duality methods for analyzing multi-model systems," arXiv:1604.04647

Consistency: Discretizing correctly

Discretization of functions

$$C^k(X,Y) \longrightarrow \mathbb{R}^n$$

$$f \qquad\qquad (f(x_1),\ldots,f(x_n))$$

Discretization of functions

$$\mathbb{R}^m \longrightarrow C^k(X,Y) \longrightarrow \mathbb{R}^n$$

$$(a_1,\ldots,a_m) \qquad f = \sum a_i f_i(x)$$

Why discretize?

$$\begin{tikzcd} \mathbb{R}^m \arrow[r] & C^k(X,Y) \arrow[r] \arrow[d, "\text{differential operator}"] & \mathbb{R}^n \\ & C^{k-p}(X,Y) & \end{tikzcd}$$

Why discretize?

$$\begin{tikzcd}
\mathbb{R}^m \arrow[r] & C^k(X,Y) \arrow[r] \arrow[d, "\text{differential operator}"] & \mathbb{R}^n \\
\mathbb{R}^m \arrow[r] & C^{k-p}(X,Y) \arrow[r] & \mathbb{R}^n
\end{tikzcd}$$

Why discretize?

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\mathbb{R}^m \arrow[r] \arrow[d, dashed, "\begin{array}{l}\text{finite}\\ \text{element}\\ \text{approx}\end{array}"'] \arrow[dr, phantom, "\not\circlearrowright"] & C^k(X,Y) \arrow[r] \arrow[d, "\begin{array}{l}\text{differential}\\ \text{operator}\end{array}"] \arrow[dr, phantom, "\not\circlearrowright"] & \mathbb{R}^n \arrow[d, dashed, "\begin{array}{l}\text{finite}\\ \text{difference}\\ \text{approx}\end{array}"] \\
\mathbb{R}^m \arrow[r] & C^{k-p}(X,Y) \arrow[r] & \mathbb{R}^n
\end{tikzcd}
$$

<u>Goals</u>:

1. Make the diagram commute as $m, n \to \infty$
   (*consistency* of the approximation)
2. Recover properties of the differential operator from
   the approximations (*convergence* of the approximation)

Goal: a sheaf interpretation

$$\begin{tikzcd}
\mathbb{R}^m \arrow[r] \arrow[d, dashed, "\text{finite element approx}"'] \arrow[dr, phantom, "\cancel{\circlearrowright}"] & C^k(X,Y) \arrow[r] \arrow[d, "\text{differential operator}"] \arrow[dr, phantom, "\cancel{\circlearrowright}"] & \mathbb{R}^n \arrow[d, dashed, "\text{finite difference approx}"] \\
\mathbb{R}^m \arrow[r] & C^{k-p}(X,Y) \arrow[r] & \mathbb{R}^n
\end{tikzcd}$$

$$\Updownarrow \text{ Encoding}$$

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "\text{approx sheaf morphism}"] & \mathbf{S} \arrow[r, "\text{approx sheaf morphism}"] & \mathbf{D}
\end{tikzcd}$$

sheaf sheaf sheaf

A simple example

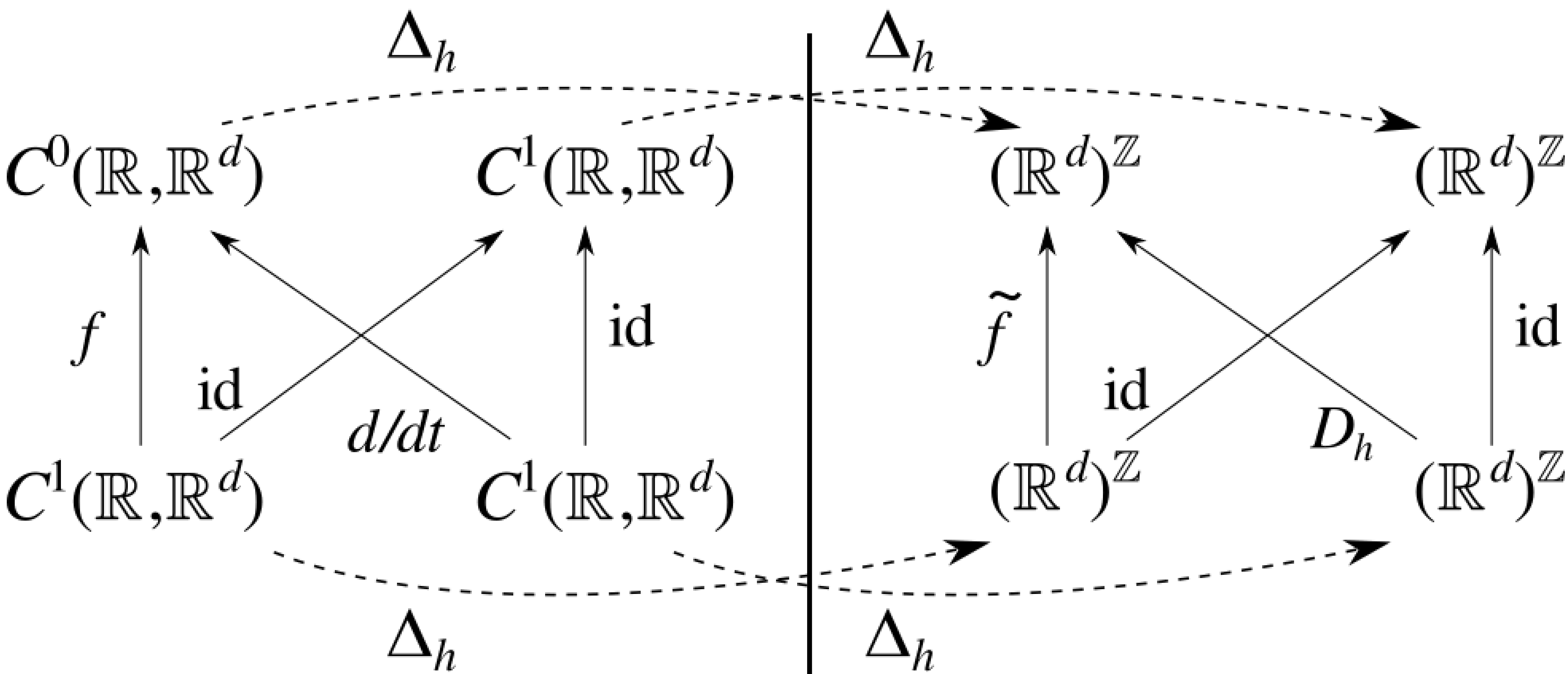
- Consider $u' = f(u)$ on the real line
- This has a sheaf diagram

$$
\begin{tikzcd}
u' & \\
C^0(\mathbb{R},\mathbb{R}^d) & C^1(\mathbb{R},\mathbb{R}^d) \\
C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "f"] \arrow[ur, "\mathrm{id}"' near start] & C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "\mathrm{id}"'] \arrow[ul, "d/dt" near start]
\end{tikzcd}
$$

Finite differences

- Discretizing each function space via a fixed step h

$$(\Delta_h u)_n = u(nh)$$



Continuous sheaf model | Discretized sheaf model

Is it a *sheaf morphism*?

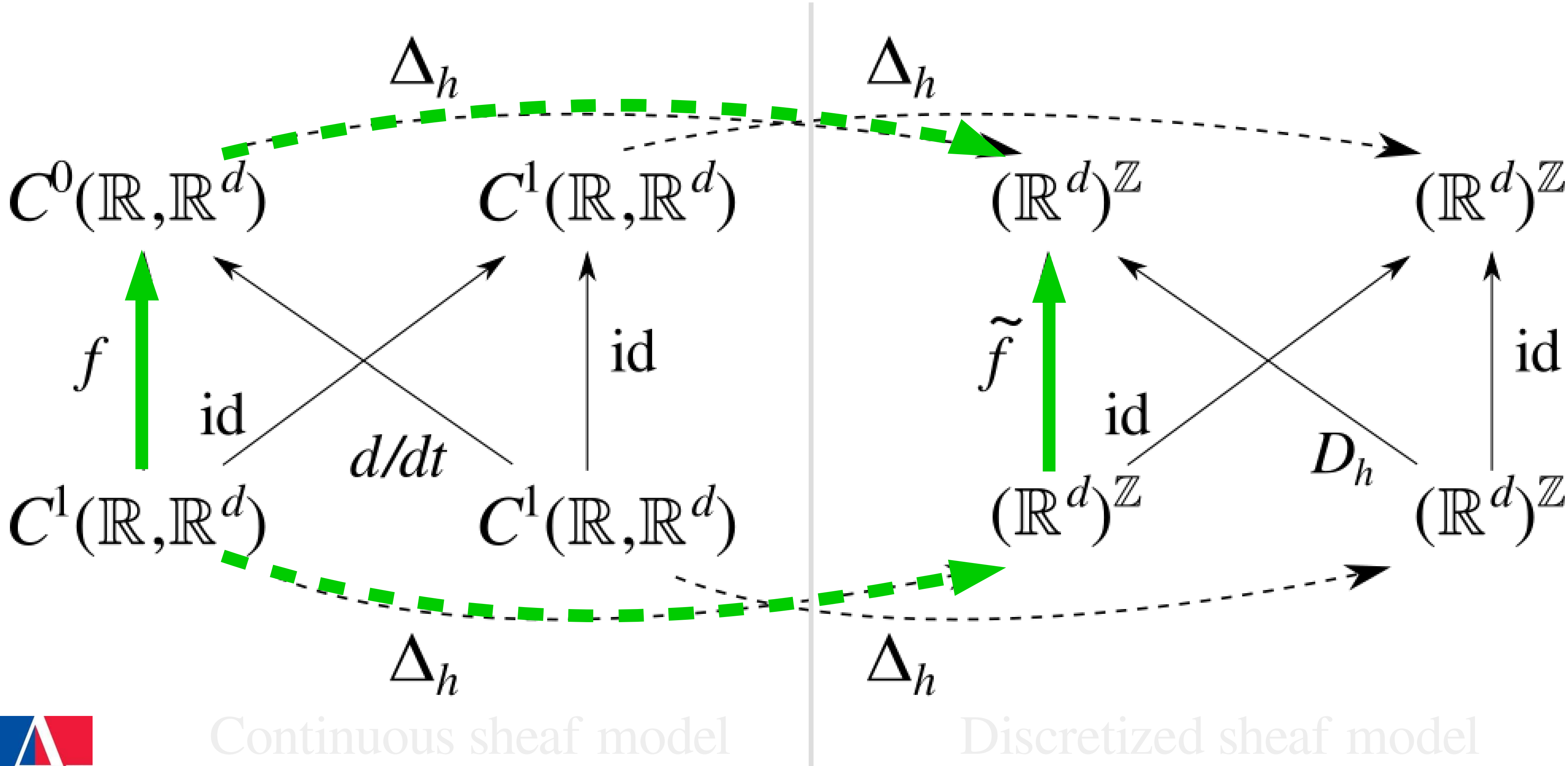
- A *sheaf morphism* is a commutative diagram specified by the dotted arrows… is this one?

$$
\begin{tikzcd}
C^0(\mathbb{R},\mathbb{R}^d) \arrow[rr, dashed, bend left, "\Delta_h"] & C^1(\mathbb{R},\mathbb{R}^d) \arrow[rr, dashed, bend left, "\Delta_h"] & (\mathbb{R}^d)^{\mathbb{Z}} & (\mathbb{R}^d)^{\mathbb{Z}} \\
C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "f"] \arrow[ur, "\mathrm{id}"] \arrow[rr, dashed, bend right, "\Delta_h"'] & C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "\mathrm{id}"'] \arrow[ul, "d/dt"] \arrow[rr, dashed, bend right, "\Delta_h"'] & (\mathbb{R}^d)^{\mathbb{Z}} \arrow[u, "\tilde{f}"] \arrow[ur, "\mathrm{id}"] & (\mathbb{R}^d)^{\mathbb{Z}} \arrow[u, "\mathrm{id}"'] \arrow[ul, "D_h"]
\end{tikzcd}
$$

Continuous sheaf model | Discretized sheaf model

Is it a *sheaf morphism*?

- This square commutes if we pick $\tilde{f}$ correctly...



Is it a *sheaf morphism*?

- … this one commutes trivially …

$$
\begin{tikzcd}
C^0(\mathbb{R},\mathbb{R}^d) \arrow[rr, bend left, dashed, "\Delta_h"] & C^1(\mathbb{R},\mathbb{R}^d) \arrow[rr, bend left, dashed, "\Delta_h"] & (\mathbb{R}^d)^{\mathbb{Z}} & (\mathbb{R}^d)^{\mathbb{Z}} \\
C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "f"] \arrow[ur, "\mathrm{id}"] \arrow[rr, bend right, dashed, "\Delta_h"'] & C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "\mathrm{id}"'] \arrow[ul, "d/dt"'] \arrow[rr, bend right, dashed, "\Delta_h"'] & (\mathbb{R}^d)^{\mathbb{Z}} \arrow[u, "\tilde{f}"] \arrow[ur, "\mathrm{id}"] & (\mathbb{R}^d)^{\mathbb{Z}} \arrow[u, "\mathrm{id}"'] \arrow[ul, "D_h"']
\end{tikzcd}
$$

Continuous sheaf model

Discretized sheaf model

Is it a *sheaf morphism*?

- … this one also commutes trivially …

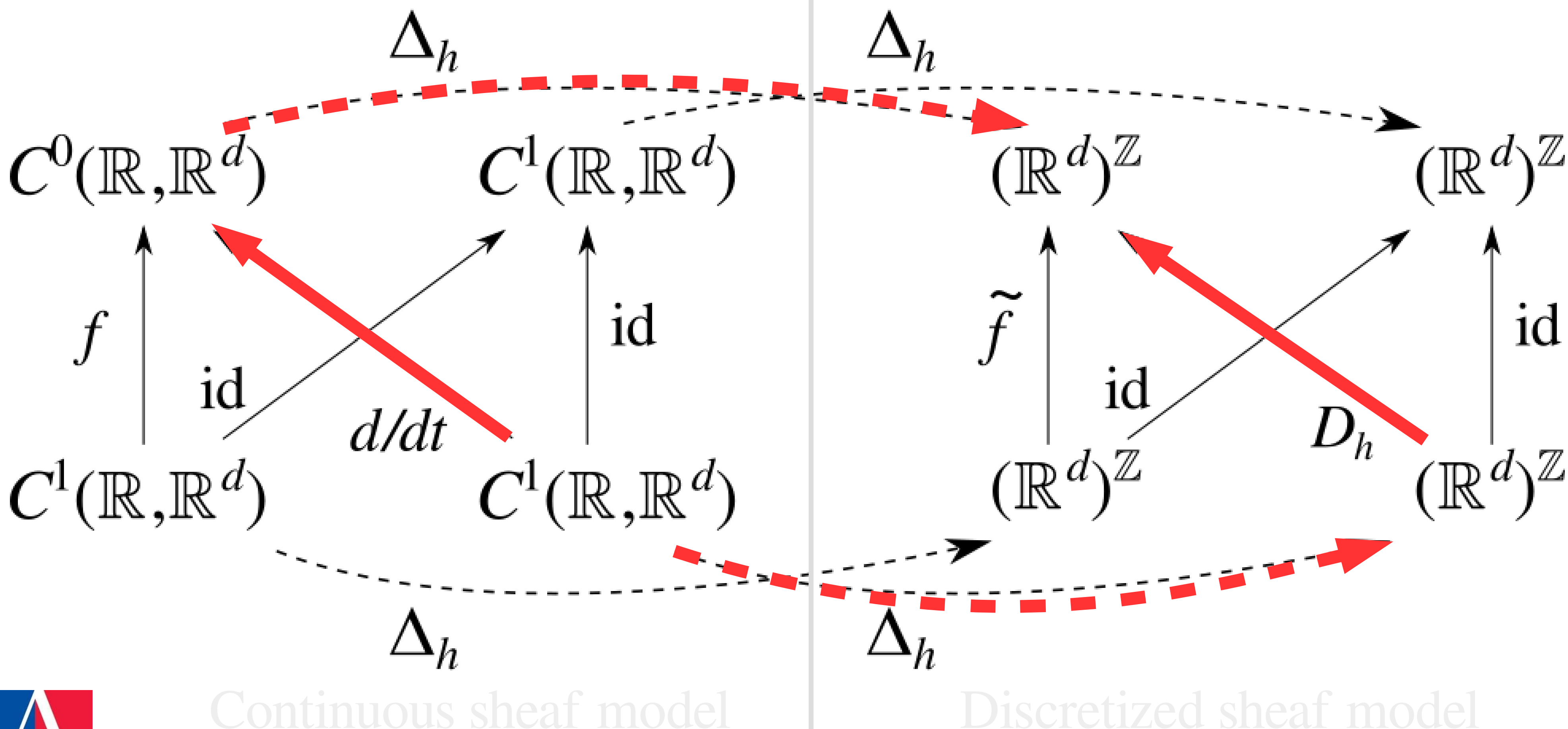
$$
\begin{tikzcd}
C^0(\mathbb{R},\mathbb{R}^d) \arrow[rr, bend left, dashed, "\Delta_h"] & C^1(\mathbb{R},\mathbb{R}^d) \arrow[rr, bend left, dashed, "\Delta_h"] & (\mathbb{R}^d)^{\mathbb{Z}} & (\mathbb{R}^d)^{\mathbb{Z}} \\
C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "f"] \arrow[ur, "\mathrm{id}"] \arrow[rr, bend right, dashed, "\Delta_h"'] & C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "\mathrm{id}"'] \arrow[ul, "d/dt"'] \arrow[rr, bend right, dashed, "\Delta_h"'] & (\mathbb{R}^d)^{\mathbb{Z}} \arrow[u, "\tilde{f}"] \arrow[ur, "\mathrm{id}"] & (\mathbb{R}^d)^{\mathbb{Z}} \arrow[u, "\mathrm{id}"'] \arrow[ul, "D_h"']
\end{tikzcd}
$$

Continuous sheaf model

Discretized sheaf model

Is it a *sheaf morphism*?

- ...but this asks that $u'(nh) = D_h u_n$, which means discretized version is **exactly correct**. Oops!



Finite elements

- We can also try to construct a finite elements approximation… from the "other side"
- Again start with the same continuous sheaf model

$$
\begin{tikzcd}
C^0(\mathbb{R},\mathbb{R}^d) & C^1(\mathbb{R},\mathbb{R}^d) \\
C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "f"] \arrow[ur, "\mathrm{id}"'] & C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "\mathrm{id}"'] \arrow[ul, "d/dt"]
\end{tikzcd}
$$

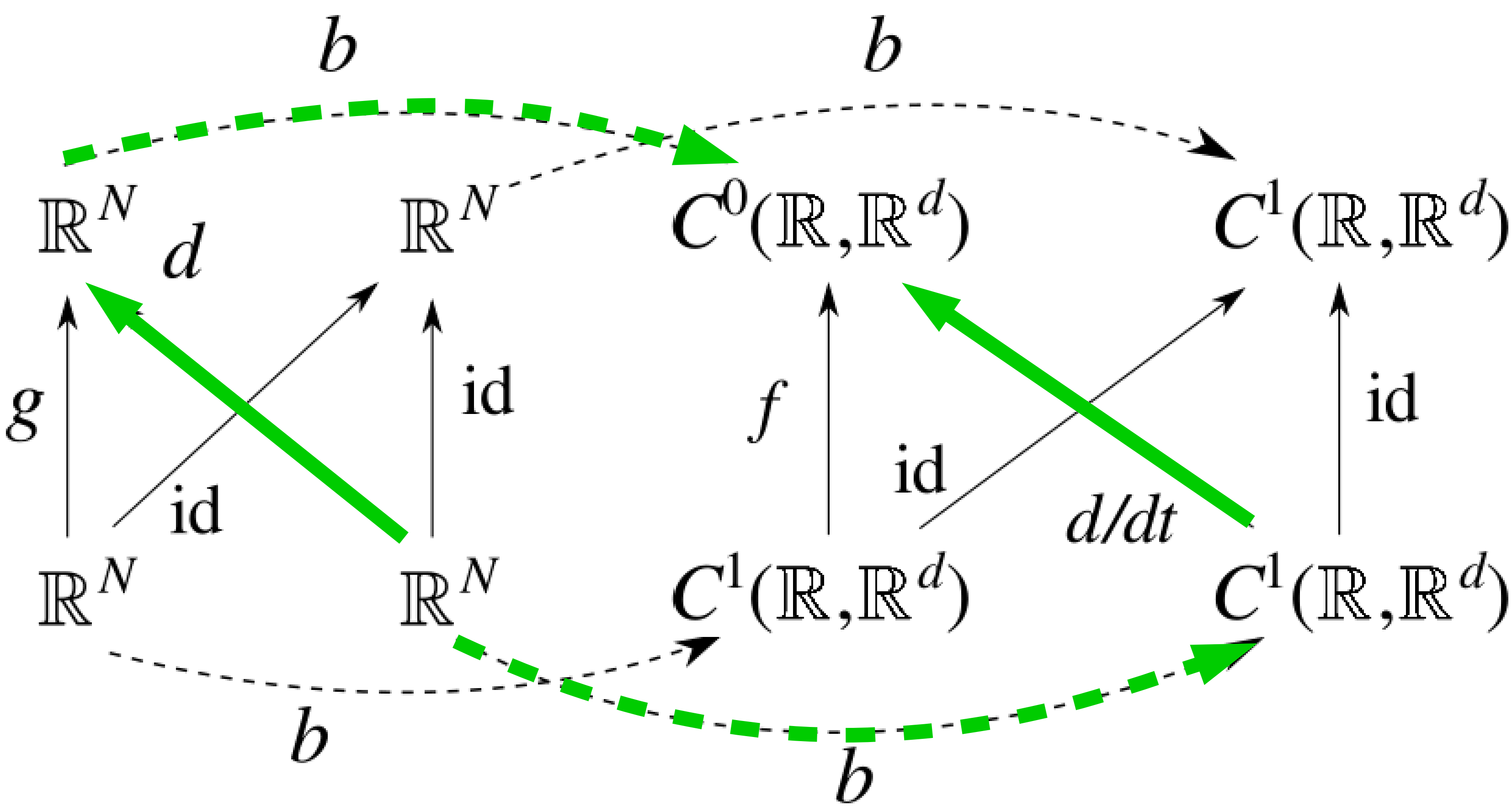
Finite elements sheaf model

- Use an N dimensional subspace of functions with a linear embedding $b : \mathbb{R}^N \to B \subseteq C^1(\mathbb{R}, \mathbb{R}^d)$.

$$
\begin{tikzcd}
\mathbb{R}^N & \mathbb{R}^N & C^0(\mathbb{R},\mathbb{R}^d) & C^1(\mathbb{R},\mathbb{R}^d) \\
\mathbb{R}^N \arrow[u, "g"] \arrow[ur, "id"' near start] \arrow[rr, dashed, bend right, "b"'] & \mathbb{R}^N \arrow[u, "id"'] \arrow[ul, "d"' near end] \arrow[rr, dashed, bend right, "b"'] & C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "f"] \arrow[ur, "id" near start] & C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "id"'] \arrow[ul, "d/dt" near start]
\arrow[from=1-1, to=1-3, dashed, bend left, "b"]
\arrow[from=1-2, to=1-4, dashed, bend left, "b"]
\end{tikzcd}
$$

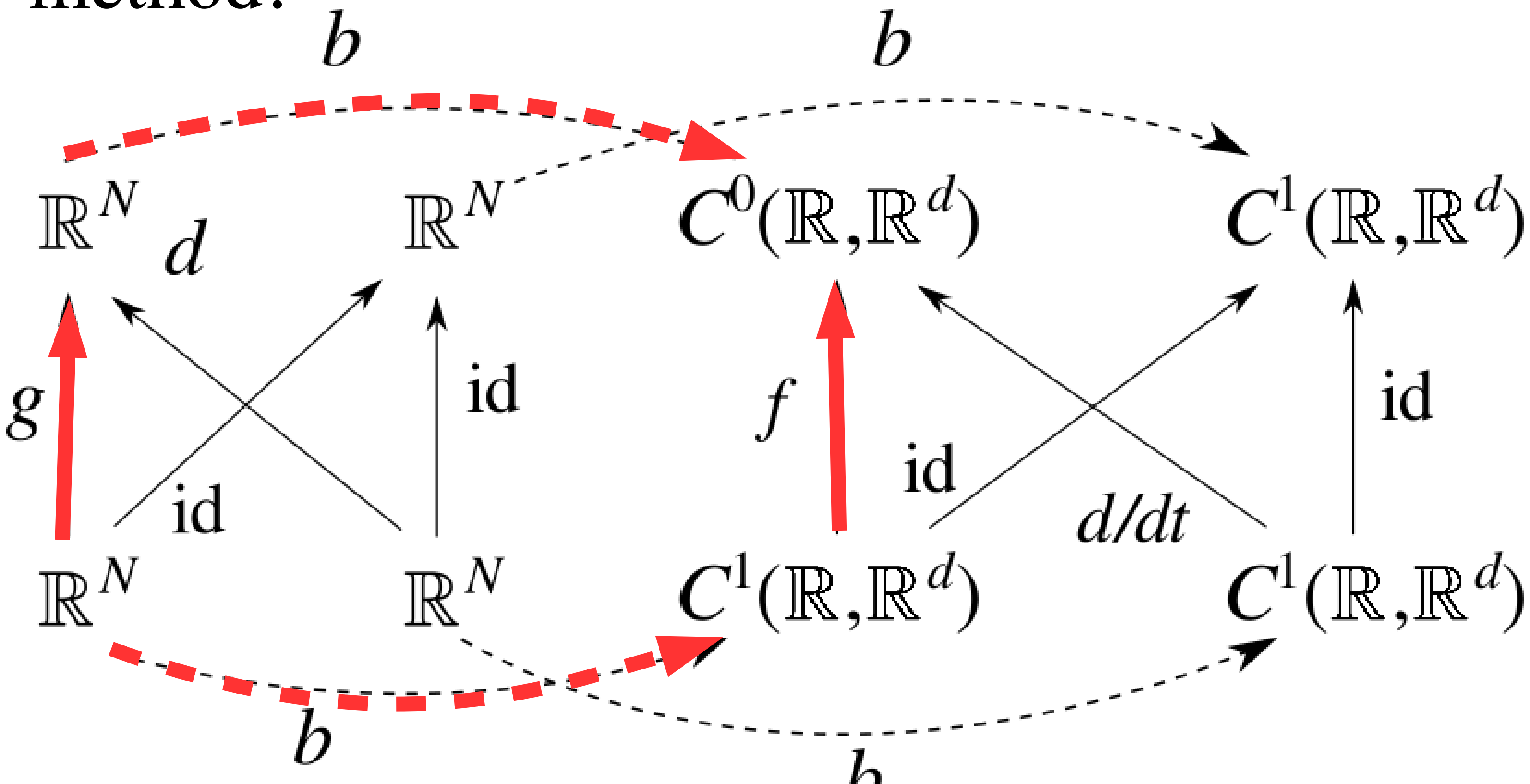
Is it a sheaf morphism?

- Although the derivative approximation can now be corrected by a judicious choice of embedding b…



Might be a sheaf morphism...

- …if not linear, now the equation itself fails
- …if linear, we may get a morphism; Galerkin method!



Observations about consistency

$$
\begin{tikzcd}
\mathbb{R}^m \arrow[r] \arrow[d, dashed, "\text{finite element approx}"'] & C^k(X,Y) \arrow[r] \arrow[d, "\text{differential operator}"] & \mathbb{R}^n \arrow[d, dashed, "\text{finite difference approx}"] \\
\mathbb{R}^m \arrow[r] & C^{k-p}(X,Y) \arrow[r] & \mathbb{R}^n
\end{tikzcd}
$$

$\Updownarrow$ Encoding

If linear, can be exact!

$$
\begin{tikzcd}
\mathbf{C} \arrow[r, "\text{approx sheaf morphism}", "\textbf{defect in approximating the equation}"'] & \mathbf{S} \arrow[r, "\text{approx sheaf morphism}", "\textbf{defect in approximating derivatives}"'] & \mathbf{D}
\end{tikzcd}
$$

sheaf | sheaf | sheaf

Convergence: Behavior of solutions

Global sections and dynamics

- Notice that $u' = f(u)$ is autonomous
- Thus global sections of are invariant under the action of time translation… can we generalize?

$$
\begin{tikzcd}
C^0(\mathbb{R},\mathbb{R}^d) \arrow[rr, dashed, bend left] & C^1(\mathbb{R},\mathbb{R}^d) \arrow[rr, dashed, bend left] & C^0(\mathbb{R},\mathbb{R}^d) & C^1(\mathbb{R},\mathbb{R}^d) \\
C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "f"] \arrow[ur, "\mathrm{id}"] \arrow[rr, dashed, bend right] & C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "\mathrm{id}"'] \arrow[ul, "d/dt"'] \arrow[rr, dashed, bend right] & C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "f"] \arrow[ur, "\mathrm{id}"] & C^1(\mathbb{R},\mathbb{R}^d) \arrow[u, "\mathrm{id}"'] \arrow[ul, "d/dt"']
\end{tikzcd}
$$

Dynamics on sheaves

- Sheaf $\mathbf{S}$ and a diffeomorphism $f : \mathbf{S}(X) \to \mathbf{S}(X)$ on its space of global sections.
- Does it extend to a sheaf automorphism?
  - In our simple example, it does!
  - In general, though, it may not!
- Conjecture: there is a cohomological obstruction
- But I do have a lead...

Sheaf dynamics theorem

- Sheaf **S** and a diffeomorphism $f : \mathbf{S}(X) \to \mathbf{S}(X)$ on its space of global sections.
- <u>Theorem:</u> (R.) There is a (possibly different) sheaf **R** with the same (or more) global sections as **S**
  - There is a sheaf morphism F: $\mathbf{S} \to \mathbf{R}$ that induces f on global sections
  - **R** is universal: any other such sheaf **P** factors through **R**

$$\begin{tikzcd} \mathbf{S} \arrow[r, "\mathrm{F}"] \arrow[dr] & \mathbf{R} \arrow[d] \\ & \mathbf{P} \end{tikzcd}$$

Proof technique: pushouts

- First, construct the stalks and component maps...

$$\begin{tikzcd}
\mathbf{S}(X) \arrow[rrr] \arrow[dr] & & & \mathbf{S}(X) \arrow[dl, dashed] \\
& \mathbf{S}(U) \arrow[r, dashed] & \mathbf{R}(U) &
\end{tikzcd}$$

Proof technique: pushouts

- … then construct the restrictions

$$\begin{tikzcd}
\mathbf{S}(X) \arrow[rrr] \arrow[dr] \arrow[ddr] & & & \mathbf{S}(X) \arrow[dl] \arrow[ddl] \\
& \mathbf{S}(U) \arrow[r] \arrow[d] & \mathbf{R}(U) \arrow[d, dashed] & \\
& \mathbf{S}(V) \arrow[r] & \mathbf{R}(V) &
\end{tikzcd}$$

Proof technique: pushouts

- … then construct the restrictions

$$
\begin{tikzcd}
\mathbf{S}(X) \arrow[rrr] \arrow[dr] \arrow[ddr] & & & \mathbf{S}(X) \arrow[dl] \arrow[ddl] \\
& \mathbf{S}(U) \arrow[r] \arrow[d] & \mathbf{R}(U) \arrow[d, dashed] & \\
& \mathbf{S}(V) \arrow[r] & \mathbf{R}(V) &
\end{tikzcd}
$$

- More technical details: gluing, universality…

Next steps: analysis!

- When does a dynamical system induce a sheaf **auto**morphism?
- Now we understand part of the diagram… but how does it all fit together?

$$
\begin{tikzcd}
\mathbf{C} \arrow[r, "\text{approx sheaf morphism}"] \arrow[d] & \mathbf{S} \arrow[r, "\text{approx sheaf morphism}"] \arrow[d, "\text{morphism}"] & \mathbf{D} \arrow[d] \\
\mathbf{?} \arrow[r] & \mathbf{R} \arrow[r] & \mathbf{?}
\end{tikzcd}
$$

- Can we push out along approximate morphisms?

For more information

Michael Robinson

michaelr@american.edu

arXiv:1604.04647

Other preprints available from my website:

http://www.drmichaelrobinson.net/