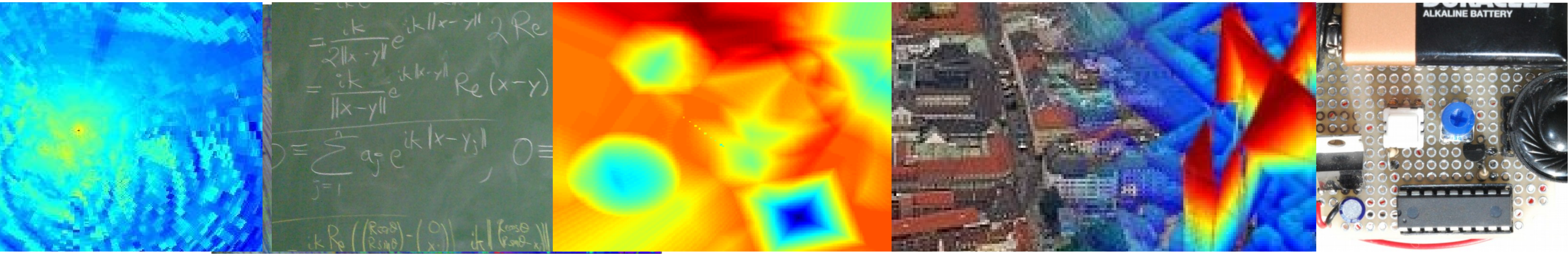
# Topological Symmetries:
## Quasiperiodicity and its Application to Filtering and Classification Problems




Michael Robinson

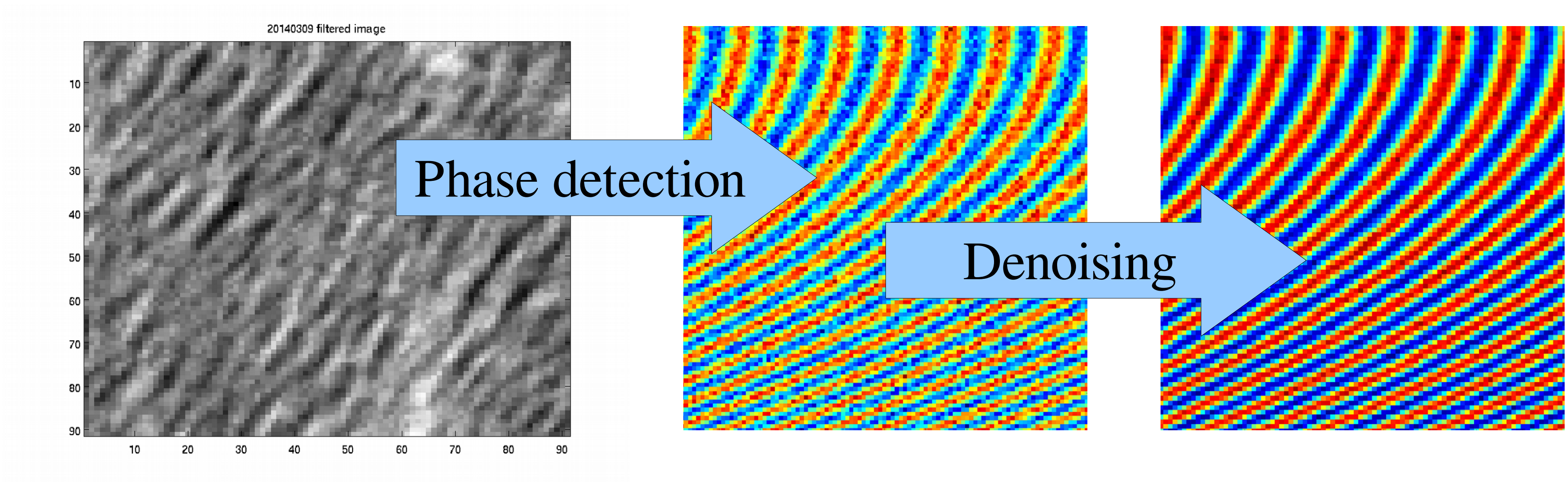
# Acknowledgements


- Collaborators:
  - Jason Summers, Charlie Gaumond (ARiA, LLC)
- Students:
  - Brian DiZio
  - Jen Dumiak
  - Sean Fennell
- Funding: Kyle Becker (ONR)

- Website reference

  http://www.drmichaelrobinson.net/

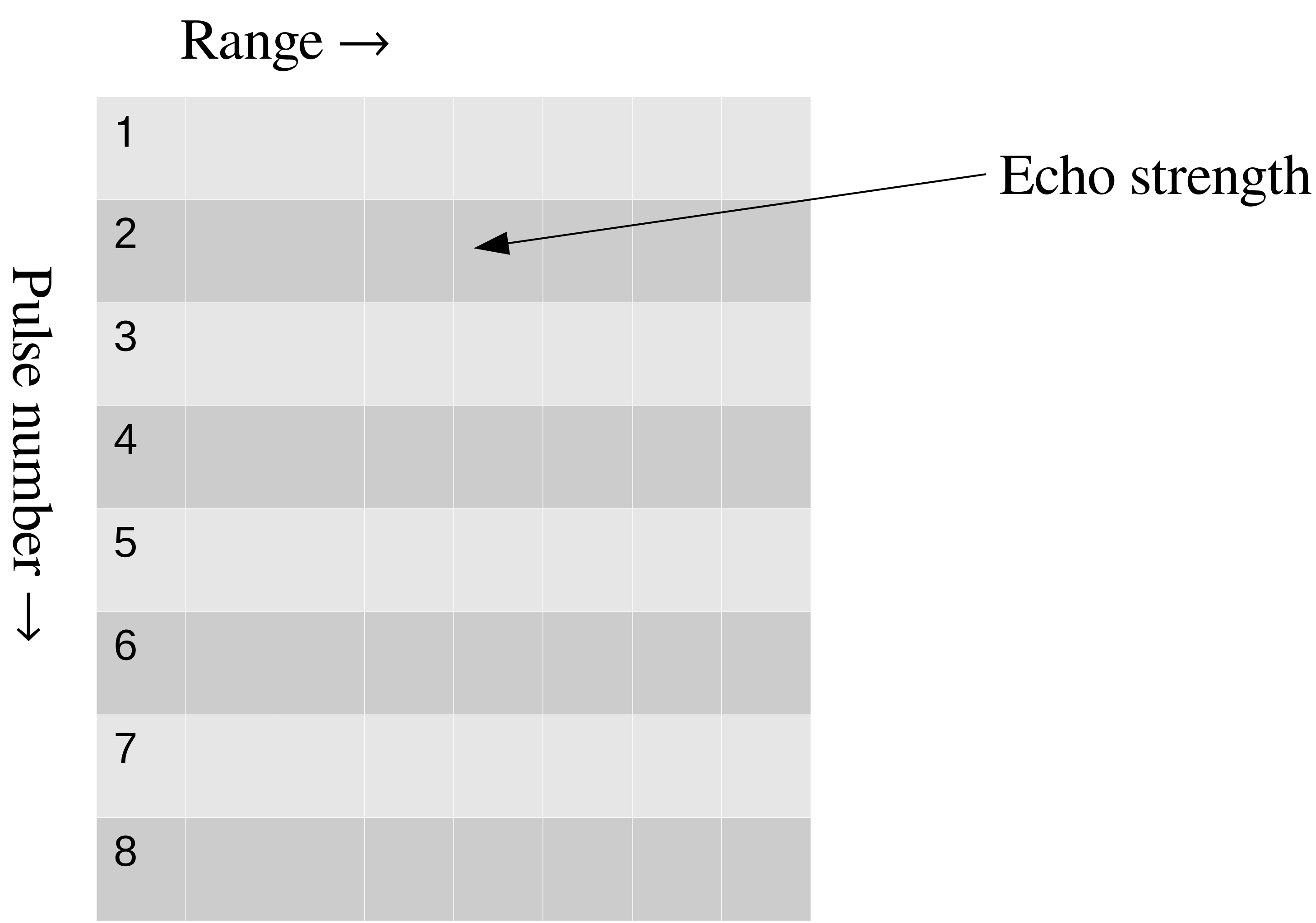
# The main idea



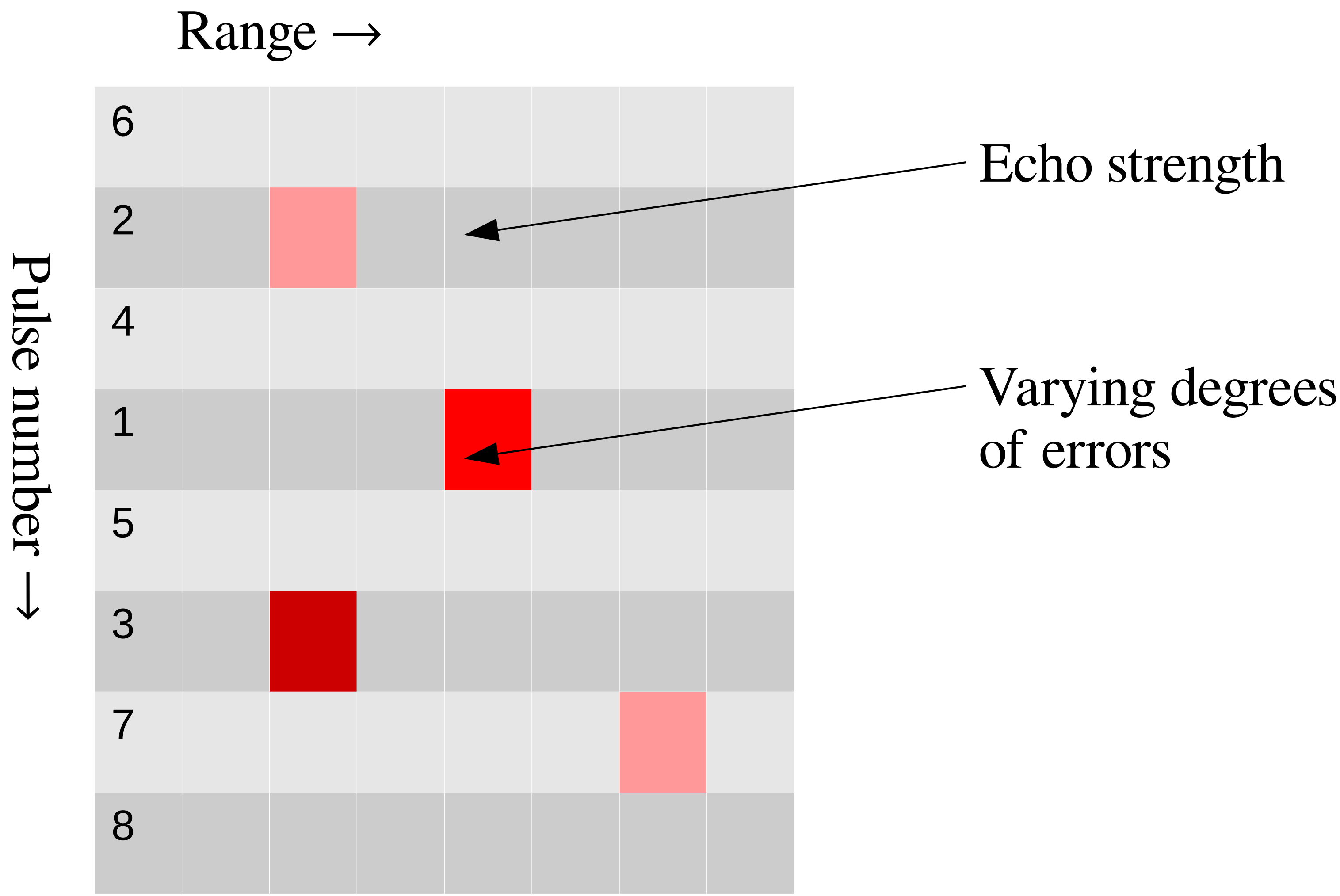

- Factoring out a smooth map from a signal may reveal a group action; **denoise on the quotient by this action**
- <u>Theorem</u>: (R.) There is an optimal, data-driven choice of domain that characterizes all symmetries in the signature

M. Robinson, "Universal factorizations of quasiperiodic functions," *SampTA* 2015, http://arxiv.org/abs/1501.06190
M. Robinson, "A Topological Lowpass Filter for Quasiperiodic Signals," *IEEE Sig. Proc. Let.*, vol. 23, no. 12, December 2016, pp. 1771-1775.

# Sonar input data format

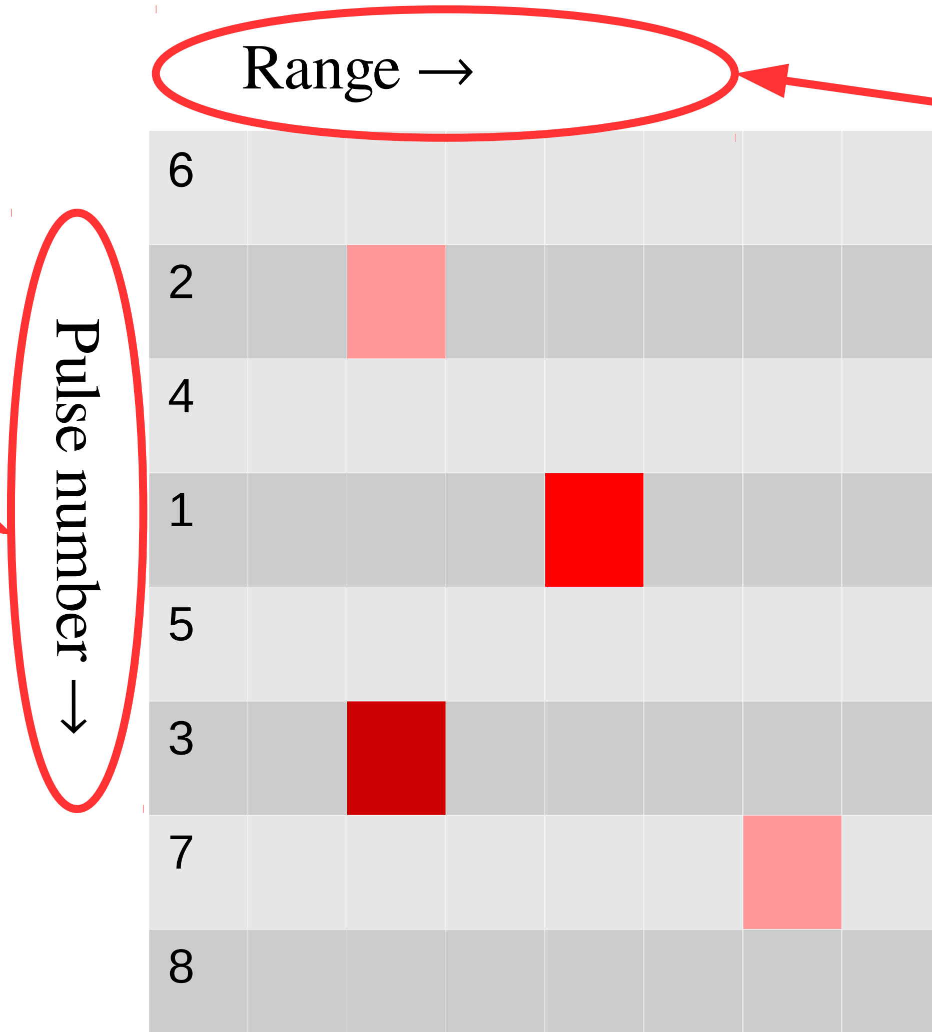


# Goal: reorganize and denoise!



# Goal: reorganize and denoise!

Partially known (but probably redundant) topology




Known geometry

# Circular coordinates

- Given $u(x)$, obtain

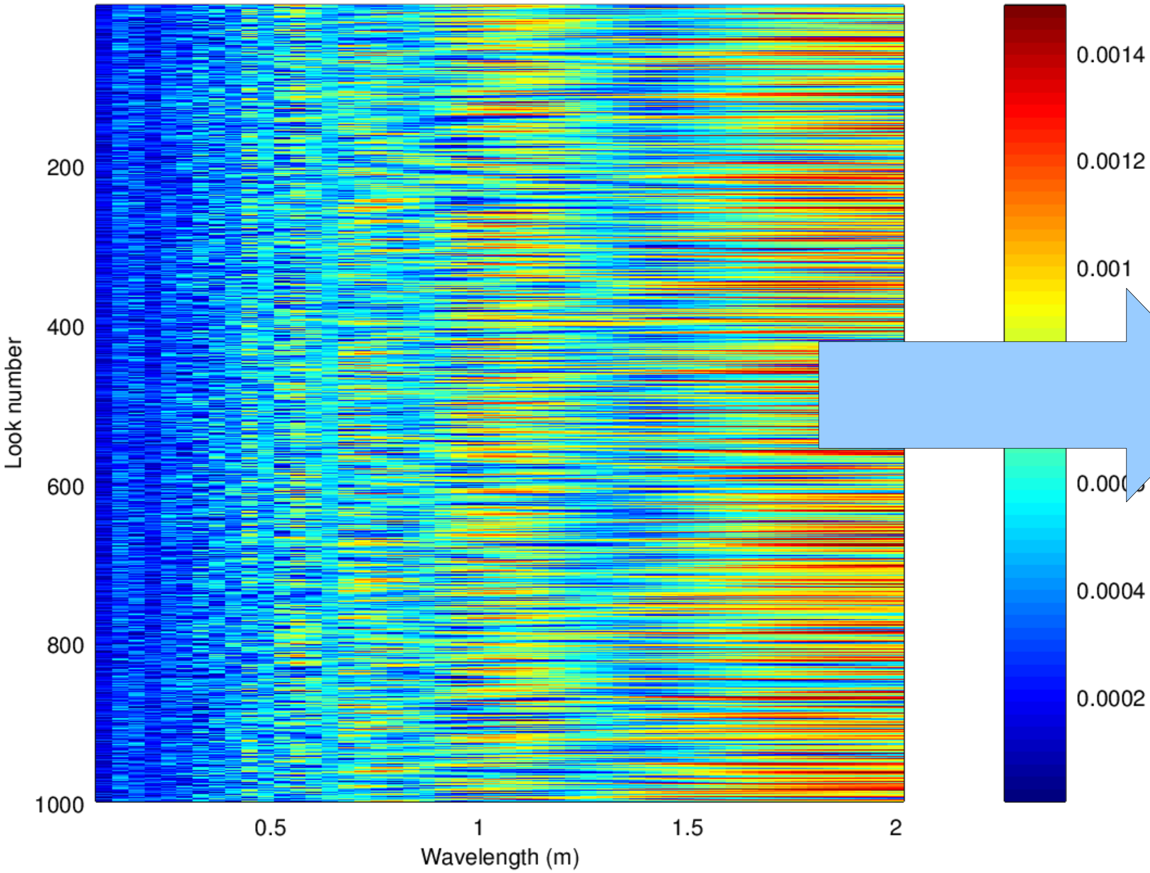
$$P(x) = (u(x), u(x + x_1), u(x + x_2), \ldots, u(x + x_n))$$

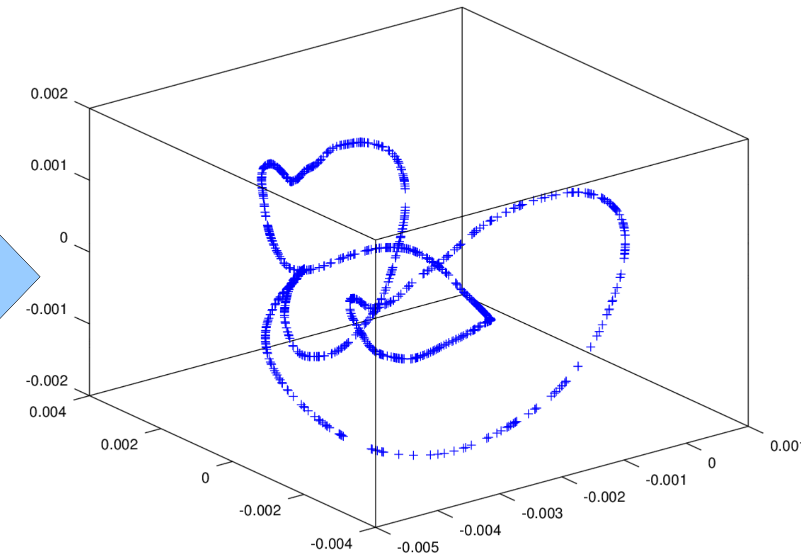
- Theorem: (Takens) Almost every $P$ is an embedding for sufficiently large $n$ and generic choice of $x_k$
- So if a function is periodic, the image of $P$ is a circle
- If $u$ is not periodic but the image of $P$ remains close to a circle (not a helix), we're still in good shape
  - Persistent cohomology* can compute smooth phase functions from time delay maps

* de Silva, Morozov, Vejdemo-Johansson "Persistent Cohomology and Circular Coordinates," *DCG* (2011).

# Topology > Pulse order in simulation

- Since the topology of the sensor space constrains the embedding, pulse order can be recovered if unknown



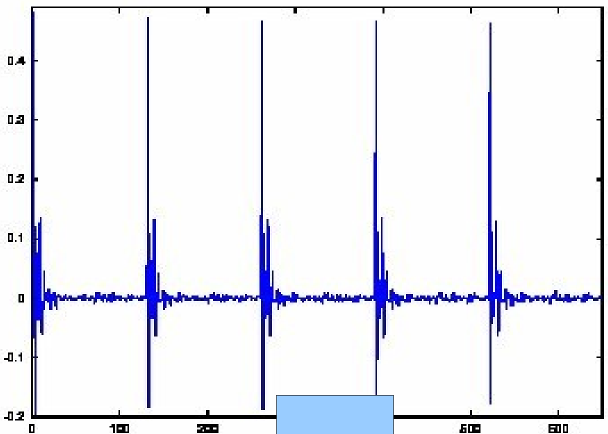




(First 3 principal components of the data)

1000 random azimuthal looks at 5 point scatterers

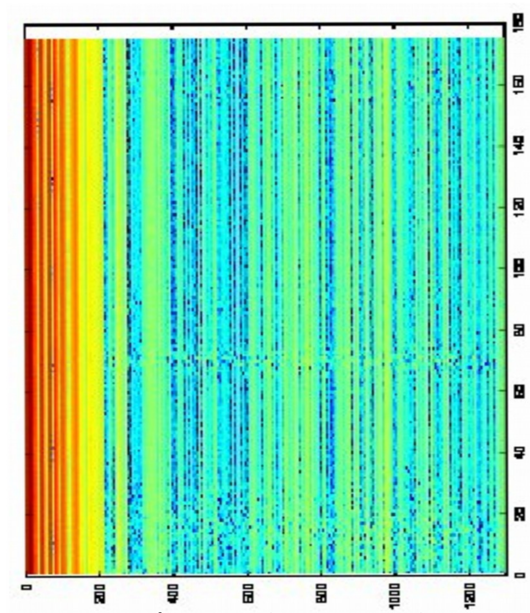
# Topology > Pulse order in practice, too

Raw time samples



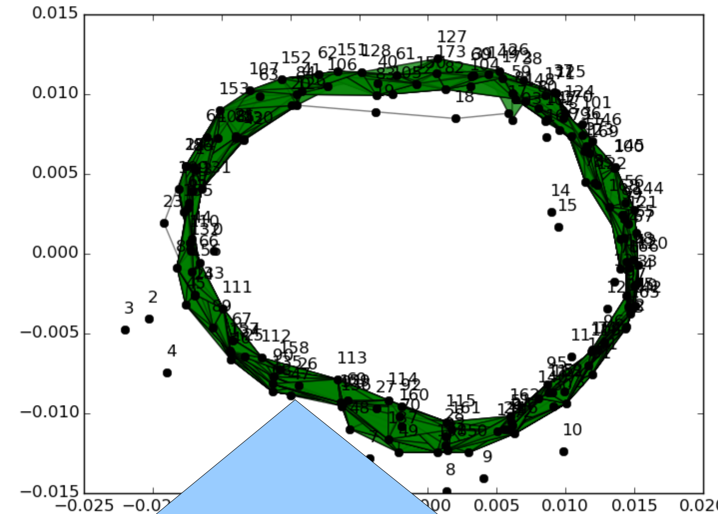
time

Pulse number



Time (Range)

Pulses organized into a phase space




M. Robinson, “Multipath-dominant, pulsed doppler analysis of rotating blades,” *IET Radar Sonar and Navigation,* Volume 7, Issue 3, March 2013, pp. 217-224.

# Re-examining periodic functions

- <u>Via symmetry</u>: A function $u$: $\mathbb{R} \to \mathbb{R}$ is *periodic* if there exists a $T$ such that

  $u(x) = u(x + T)$ for all $x$

- <u>Via diagrams</u>: Periodic functions factor through the circle:

The *phase function*
$\phi(x) = 2\pi\,[(x\,/\,T) \bmod 1]$

$$
\begin{tikzcd}
\mathbb{R} \arrow[r, "u"] \arrow[d, "\phi"'] & \mathbb{R} \\
S^1 \arrow[ur, "U"'] &
\end{tikzcd}
$$

# Re-examining periodic functions

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- Via diagrams: Periodic functions factor through the circle:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "u"] \arrow[d, "\phi"'] & \mathbb{R} \\ S^1 \arrow[ur, "U"'] & \end{tikzcd}$$

The *phase function* can be generalized...

... as can the *phase space*

# Generalizing beyond circular

- Certainly, we should ask for more general inputs and outputs... manifolds are good (want calculus)
- Then we should assume $u$, $\phi$, $U$ are all smooth

We'd like an analog of a monotonic function here: quotient map

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "\phi"'] & N \\ C \arrow[ur, "U"'] & \end{tikzcd}$$

# Why manifolds?

- If the phase space is not required to be a manifold, then the best choice is the topological quotient

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "\phi"'] & N \\ M/u \arrow[ur, "U"'] & \end{tikzcd}$$

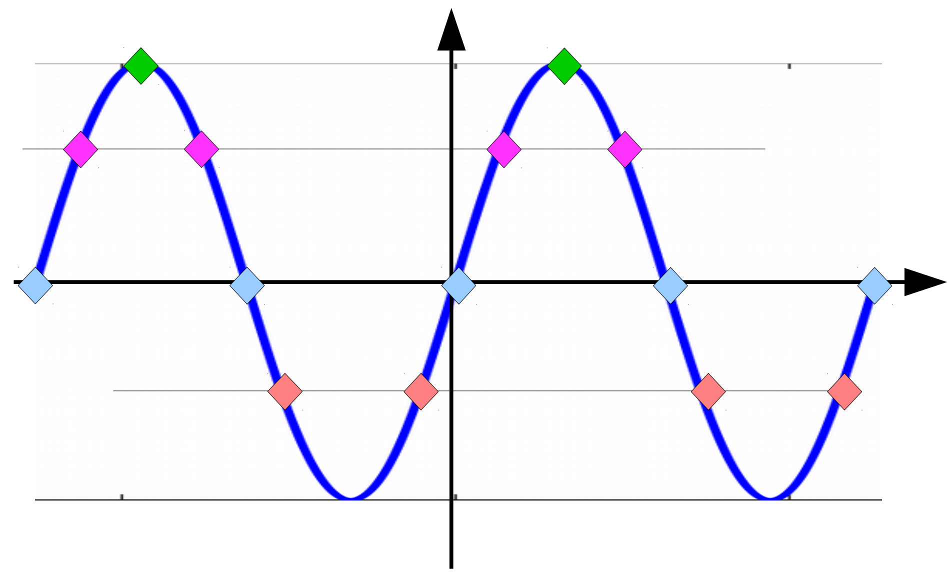
This has an annoying consequence...
$\phi$ can have critical points

# Accidental contractibility

- Problem: The phase space isn't amenable to cohomological periodicity detection using $H^1$
- Consider $u(x) = \sin x$, then the factorization looks like

$$
\begin{tikzcd}
\mathbb{R} \arrow[r, "\sin"] \arrow[d, "\phi"'] & \mathbb{R} \\
\boxed{[-\pi / 2, \pi / 2]} \arrow[ur, "U"'] &
\end{tikzcd}
$$

Contractible phase space



… so we'd better ensure $\phi$ has constant rank

# Quasiperiodicity

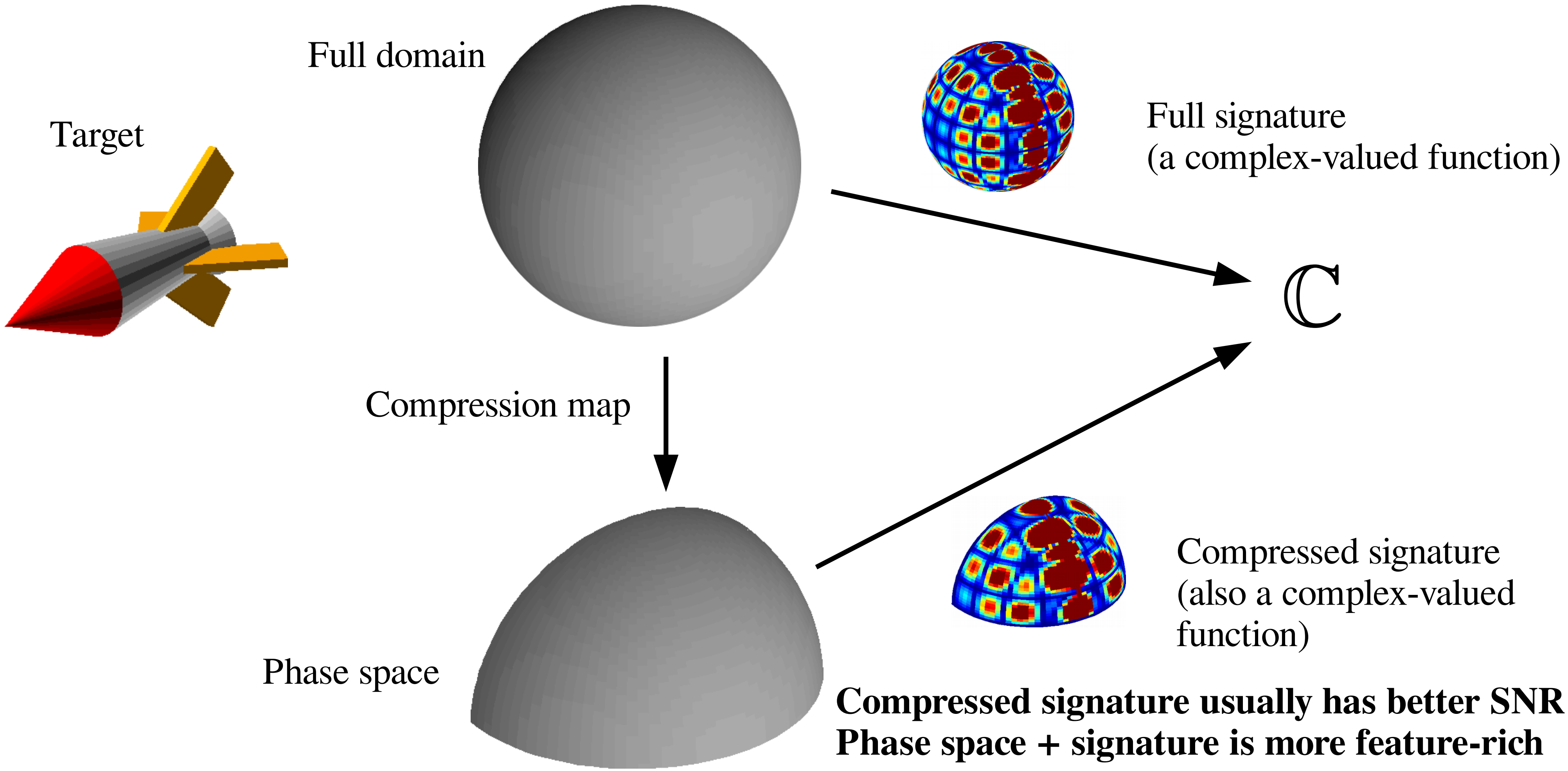
- <u>Definition</u>: a smooth function $u$ has a *quasiperiodic factorization* given by the commutative diagram below when $\phi$ is a surjective submersion

A consequence of $\phi$ being a surjective submersion is that $C$ is a manifold

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "\phi"'] & N \\ C \arrow[ur, "U"'] & \end{tikzcd}$$

- We'll say $u$ is $(\phi, C)$-*quasiperiodic* in this case

# Factorization of signatures




**Compressed signature usually has better SNR!**
**Phase space + signature is more feature-rich**

<u>Theorem</u>: Optimal compressed signatures always exist

M. Robinson, "Universal factorizations of quasiperiodic functions," *SampTA* 2015, http://arxiv.org/abs/1501.06190

# A quasiperiodic factorization

- Consider $u$: given by

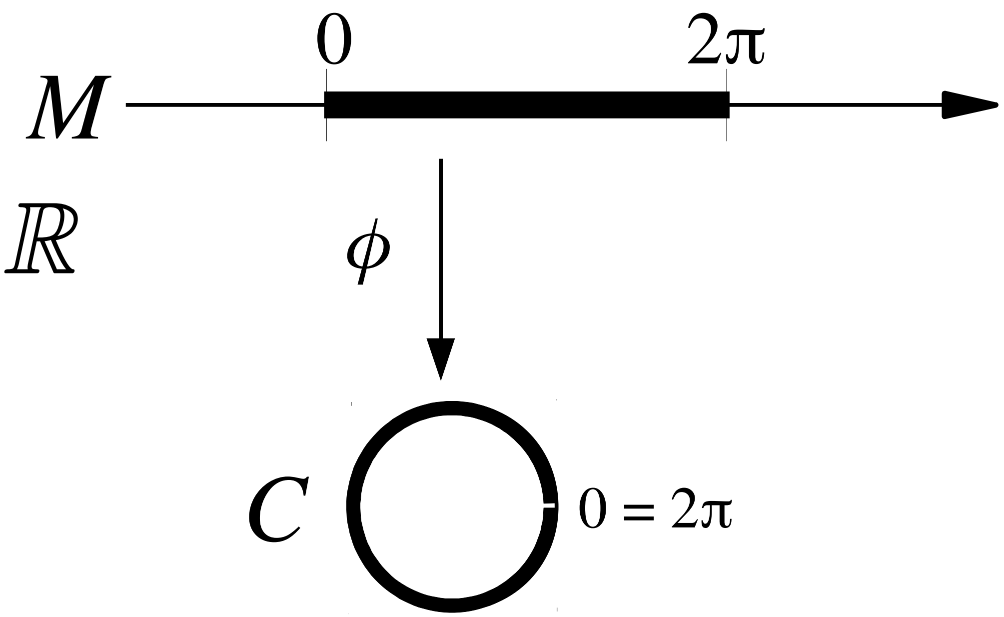
  $u(x) = \sin x$

- Here's a quasiperiodic factorization

  $\phi(x) = x \bmod 2\pi$

  $U(x) = \sin x$

$$
\begin{tikzcd}
M = \mathbb{R} \arrow[r, "u"] \arrow[d, "\phi"'] & N = \mathbb{R} \\
C = S^1 \arrow[ur, "U"'] &
\end{tikzcd}
$$



# A quasiperiodic factorization

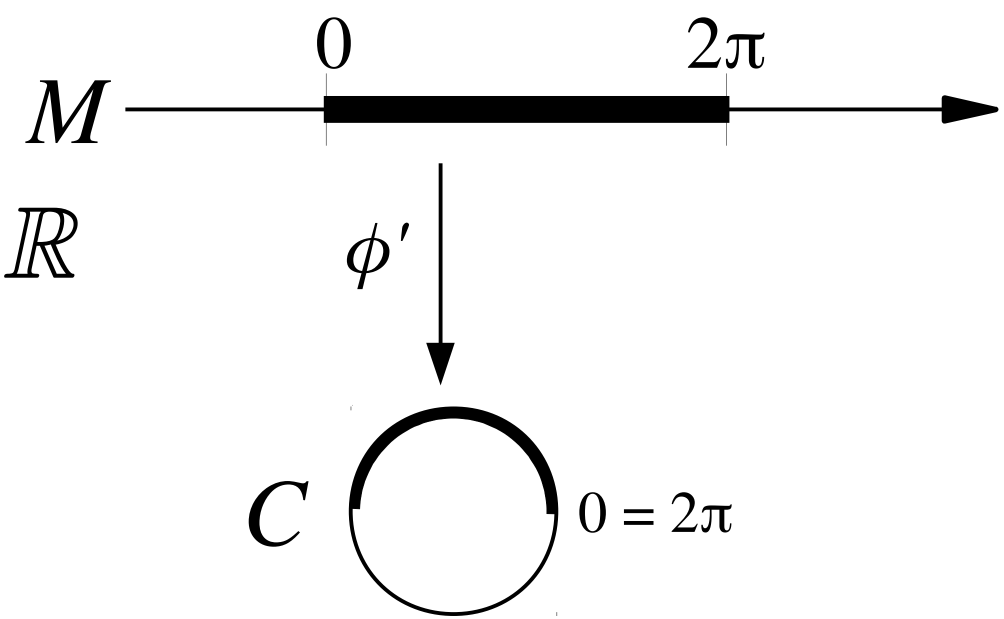
- Consider $u$: given by

  $u(x) = \sin x$

- Here's another quasiperiodic factorization

  $\phi'(x) = (x\,/\,2) \bmod 2\pi$

  $U'(x) = \sin 2x$




$$
\begin{tikzcd}
M = \mathbb{R} \arrow[r, "u"] \arrow[d, "\phi'"'] & N = \mathbb{R} \\
C = S^1 \arrow[ur, "U'"'] &
\end{tikzcd}
$$

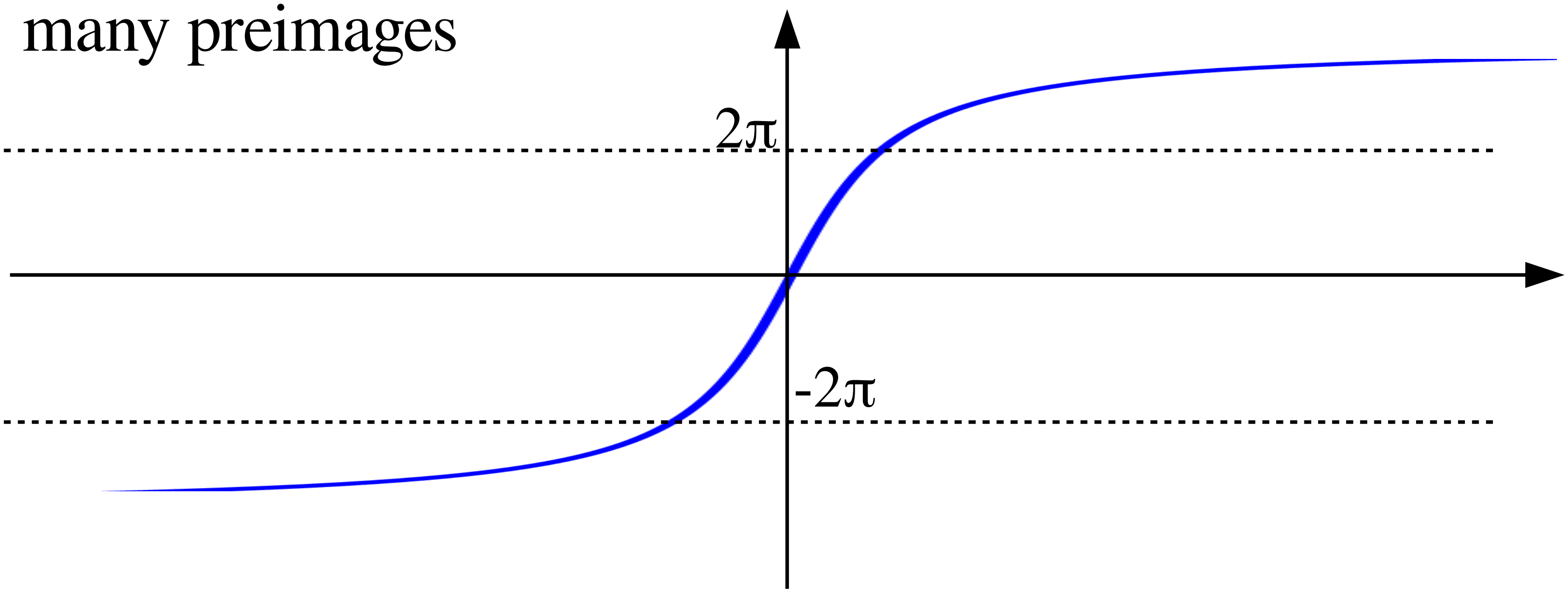
# Factorizations can be weird

- Consider $u\colon \mathbb{R}\rightarrow S^1$ given by $u = U \circ \phi$ where

  $\phi : \mathbb{R}\rightarrow S^1$, given by $\phi(x) = (6 \arctan x) \bmod 2\pi$

  $U : S^1\rightarrow S^1$, given by $U(x) = x$

  – This is a quasiperiodic factorization, but the function doesn't “repeat” – every point in the range has finitely many preimages



# Non-uniqueness of factorizations

- The category **QuasiP**$(u)$ for a smooth function $u$:
  - Objects: quasiperiodic factorizations $(\phi, U)$
  - Morphisms: $(\phi,U)\rightarrow(\phi',U')$ if there's a commutative diagram

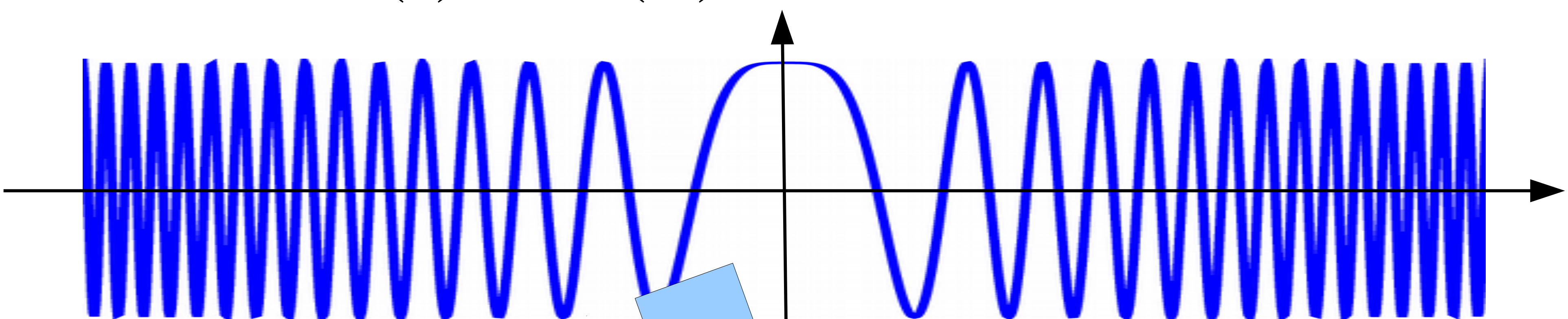
$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\phi'"'] & C \arrow[d, "U"] \arrow[dl] \\ C' \arrow[r, "U'"'] & N \end{tikzcd}$$

- <u>Theorem</u>: (R.) **QuasiP**$(u)$ has a unique final object, called the *universal quasiperiodic factorization* of $u$
  - It's the correct phase space for a topological filter tuned to find $u$ in a noisy signal

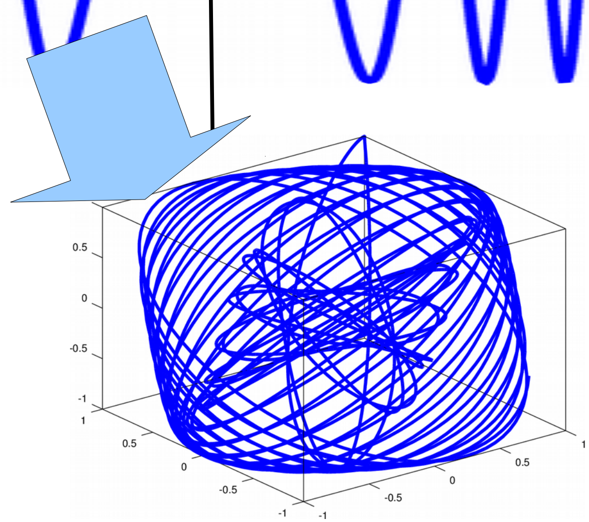
# Algorithmics:
# finding quasiperiodic factorizations

# Algorithmics: delay embeddings

- Choosing dimension of the ambient space is tricky:
  - Too high or too low dimensionality is a problem
- Consider $u(x) = \cos(x^2)$



A typical delay embedding with too low dimension gets “tangled” with self-intersections...



# Algorithmics: delay embeddings

- Choosing dimension of the ambient space is tricky:
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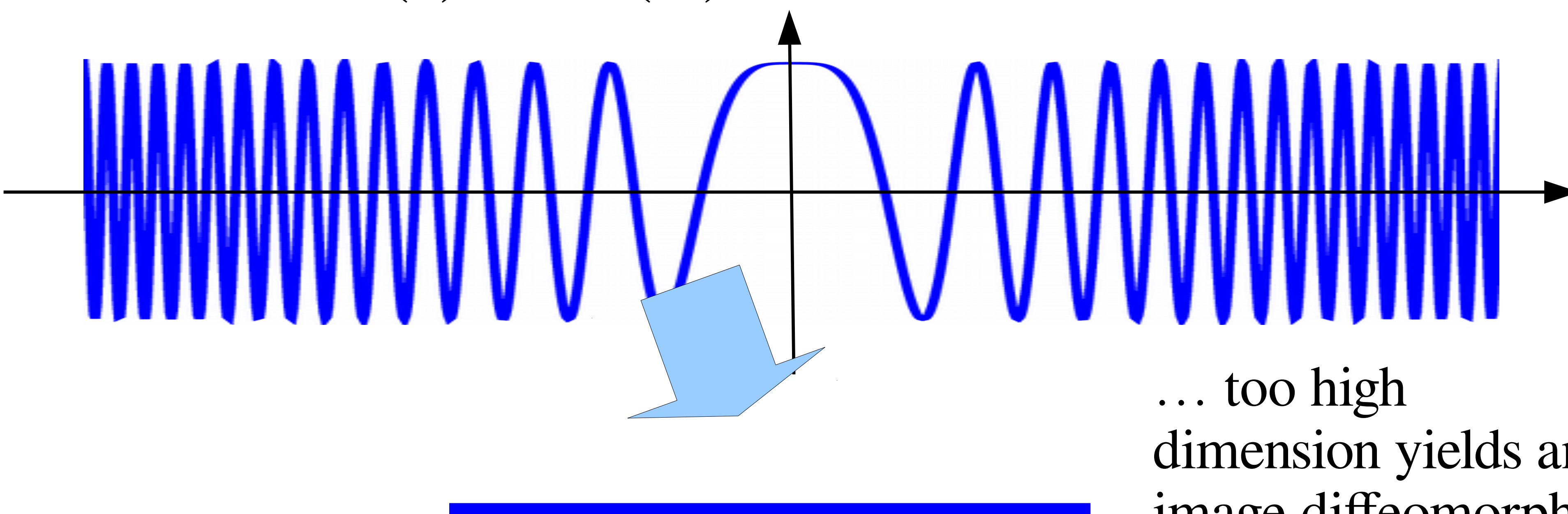
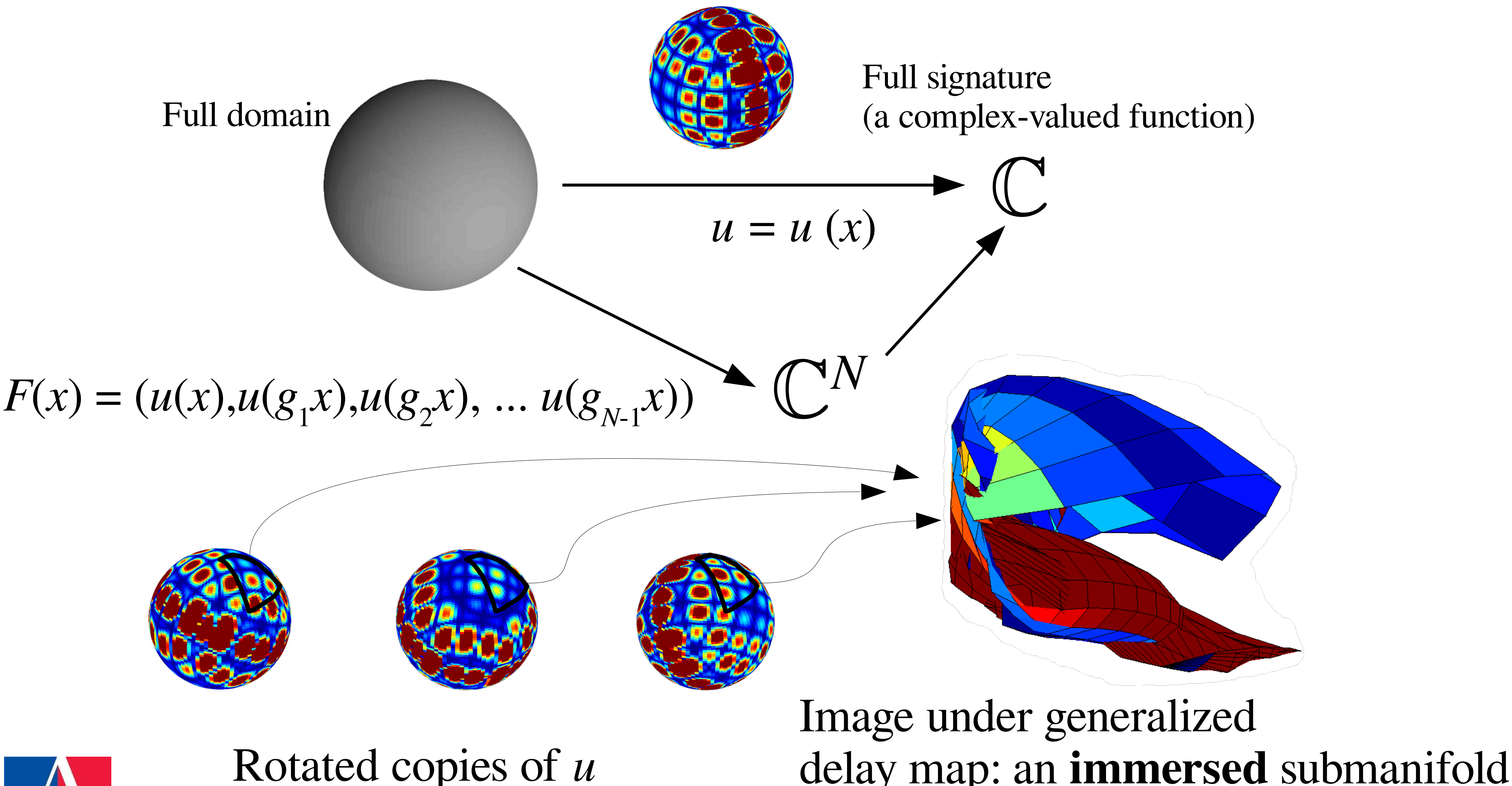


… too high dimension yields an image diffeomorphic to $\mathbb{R}$: a trivial factorization

# Topological estimation

- Generalize to group actions on manifolds!



# How does it work?

- Picking the rotations can be done, but not arbitrarily!
- Lemma: If $u : M \rightarrow N$ is a smooth map and
  - $M$ is a compact manifold
  - $G$ is a group of diffeomorphisms acting on $M$ transitively

  Then there is a finite set $\{g_1, \ldots g_m\}$ such that

  $$F(x) = (u(x), u(g_1x), .. u(g_mx))$$

  has constant rank and

  $$\text{rank } dF(x) = \text{max rank } du(y) \text{ over all } y \text{ in } M.$$

- Theorem: (R.) Use these in your generalized delay map to obtain a **universal** quasiperiodic factorization

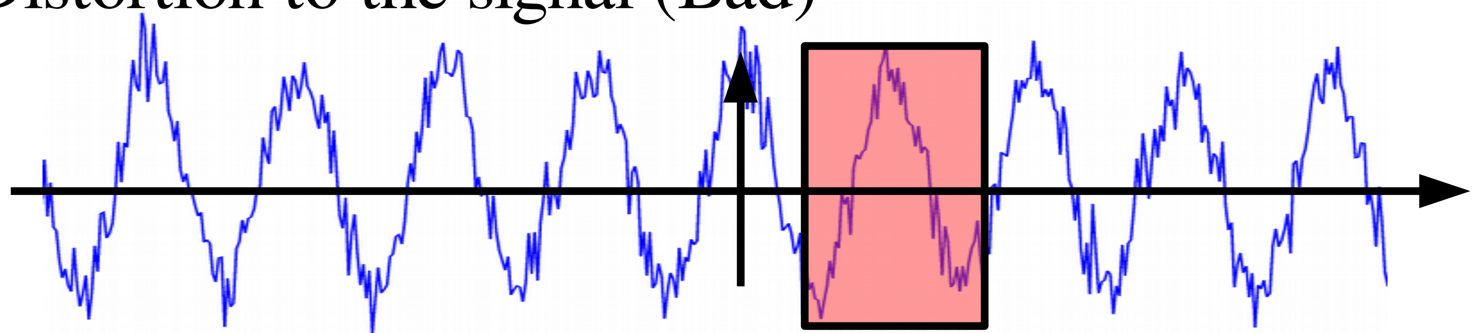
# Filtering using Quasiperiodic Factorizations

The *QuasiPeriodic Low Pass Filter*

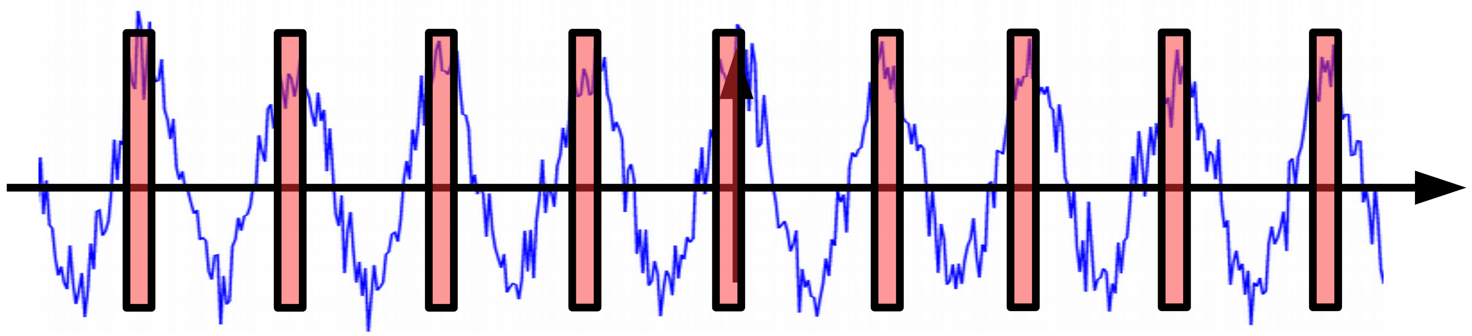
(QPLPF)

# Circumventing bandwidth limits

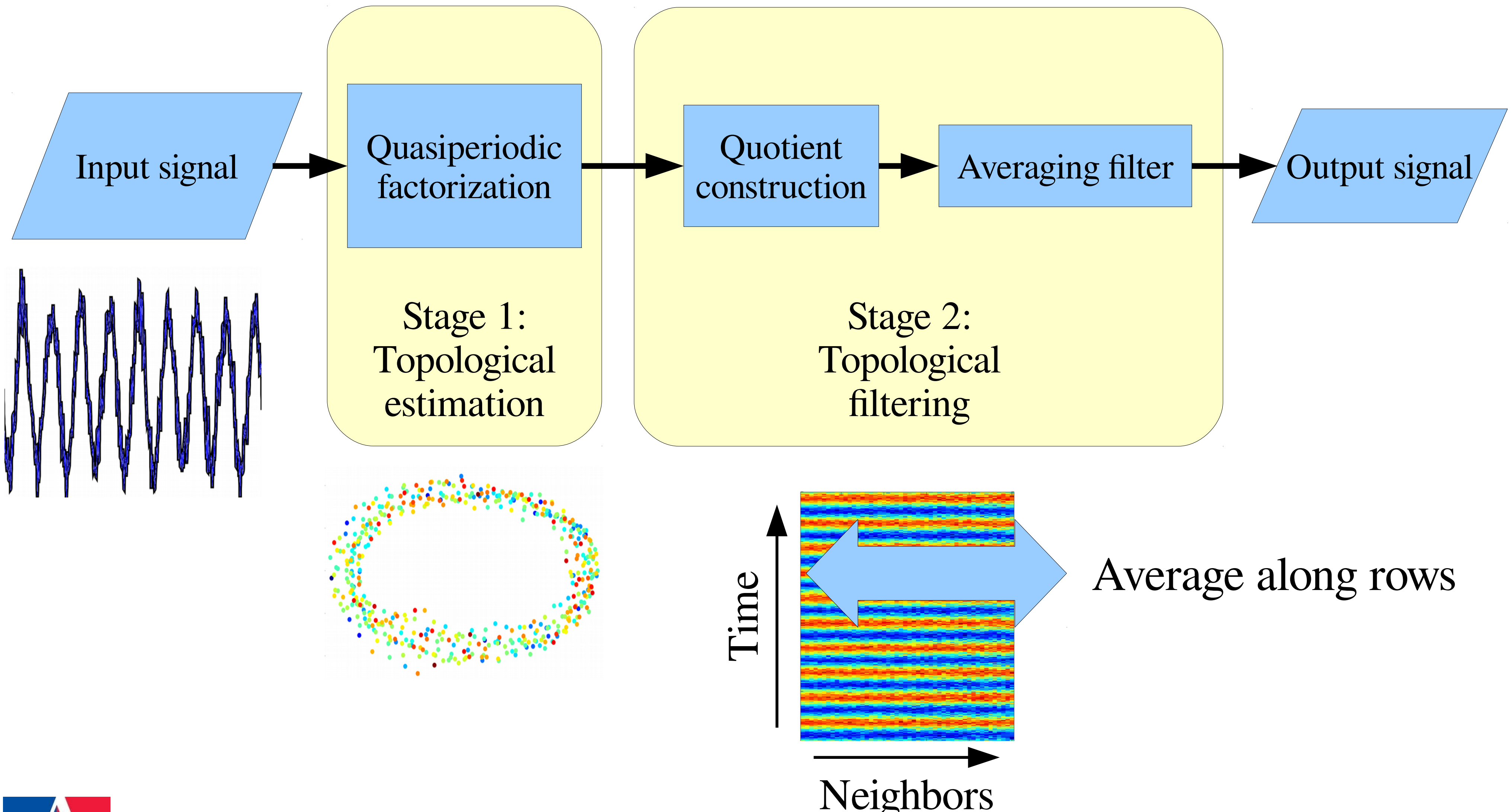
- Traditional: averaging in a connected window
  - Noise cancellation (Good)
  - Distortion to the signal (Bad)



- QPLPF: **Safely** do **more** averaging across the **entire** signal using a quasiperiodic factorization first

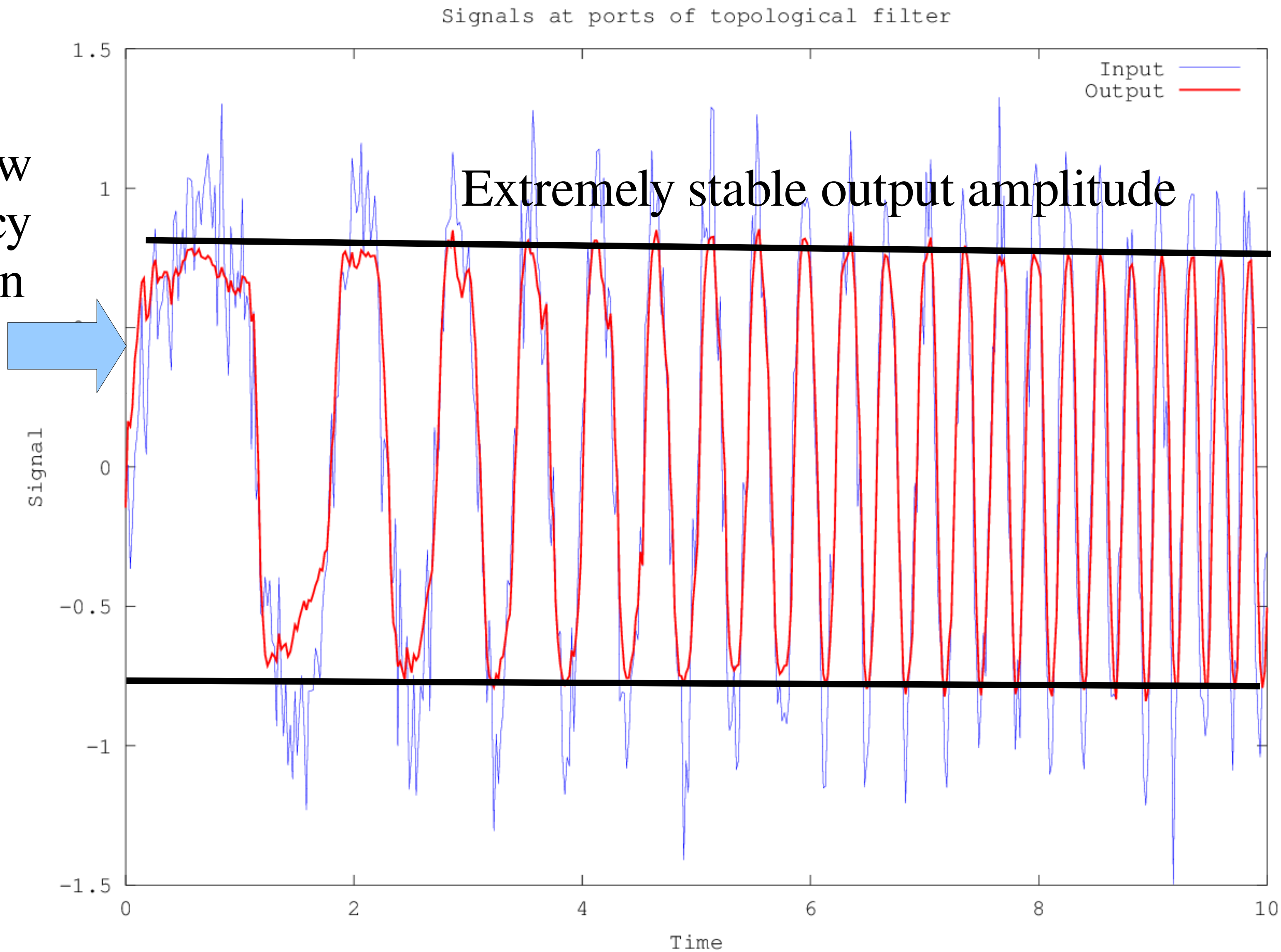


# QPLPF block diagram

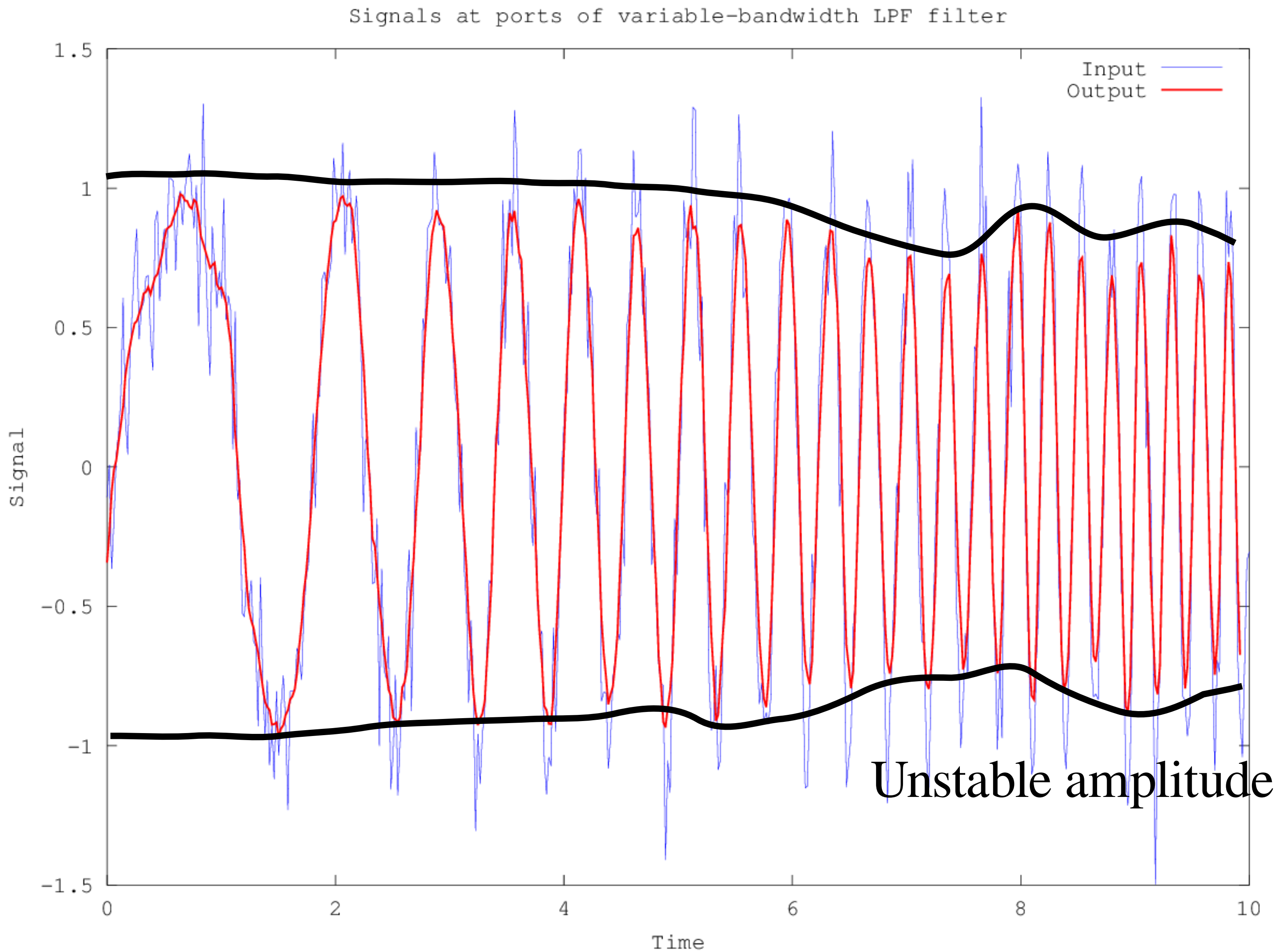


# QPLPF results

Some low frequency distortion

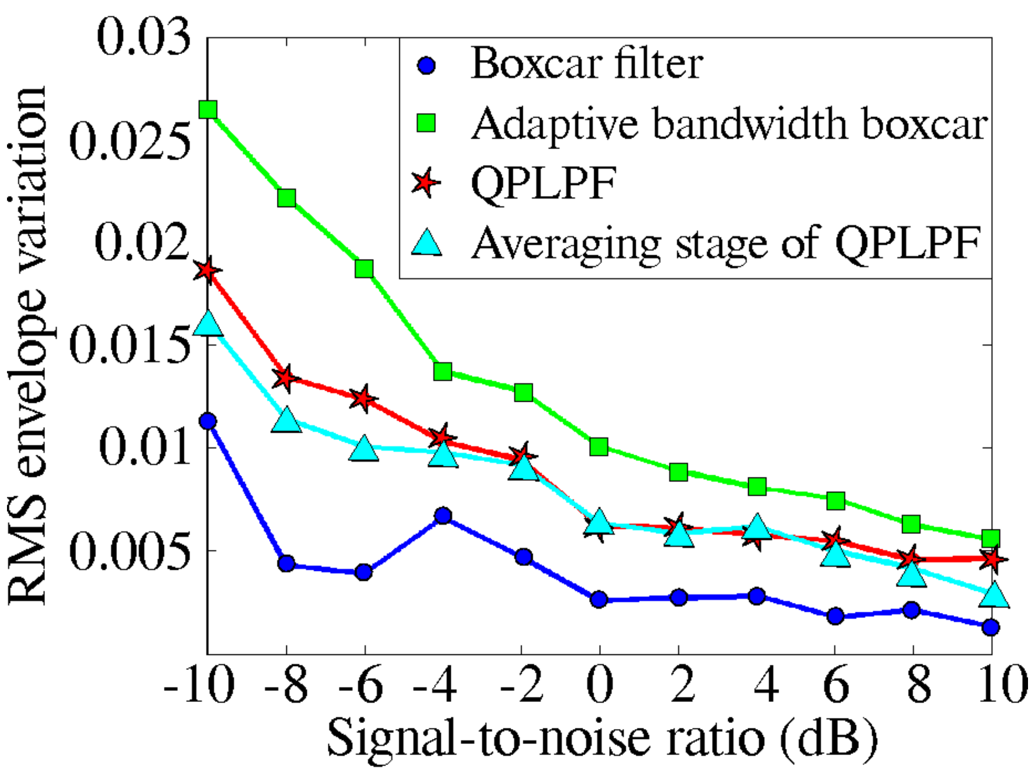


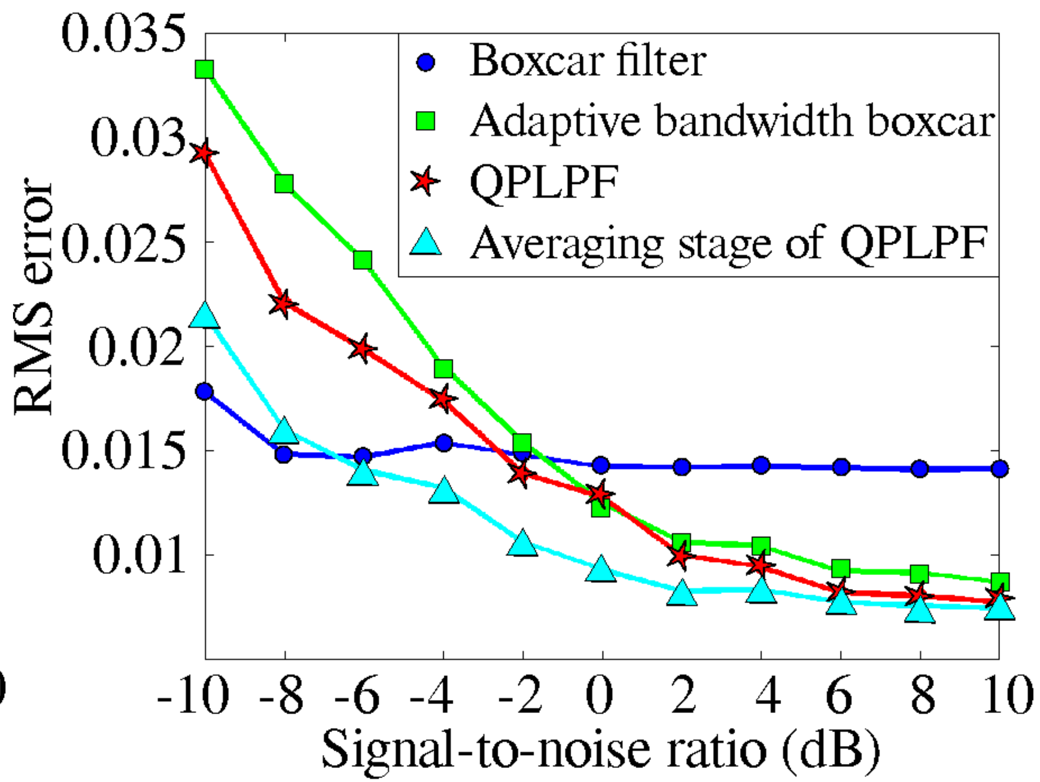
# Compare: standard adaptive filter



# Filter performance comparison

- QPLPF combines good noise removal with signal envelope stability

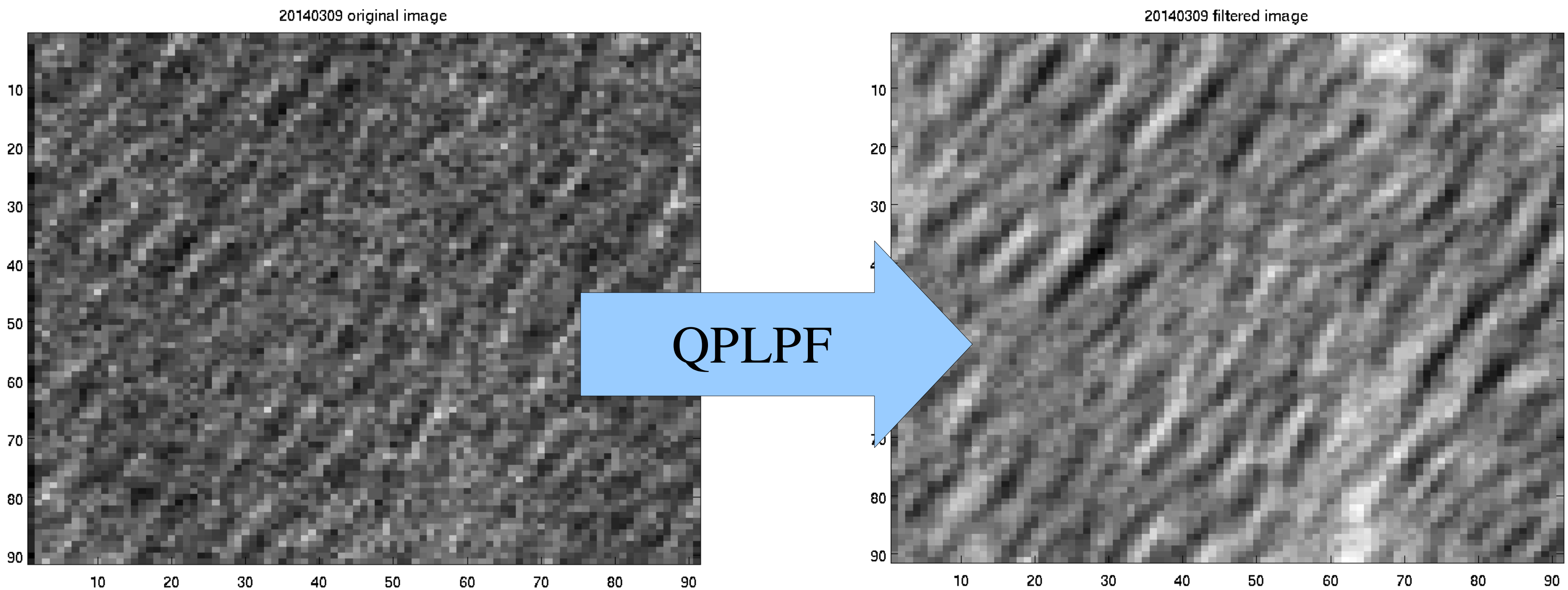


# Ocean radar image despeckling

After topological filtering:

- Speckle and contrast improved



# Next steps

- Can we find the necessary “rotation” group elements algorithmically?
  - The proof that they exist is non-constructive!
  - How many are needed practically (probably more than are required theoretically)?
- Implementations complete for dim $M = 2\ldots$ generalize!
- Already tested on ocean SAR images.. now apply to others

# For more information

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Preprints available from my website:

http://www.drmichaelrobinson.net/