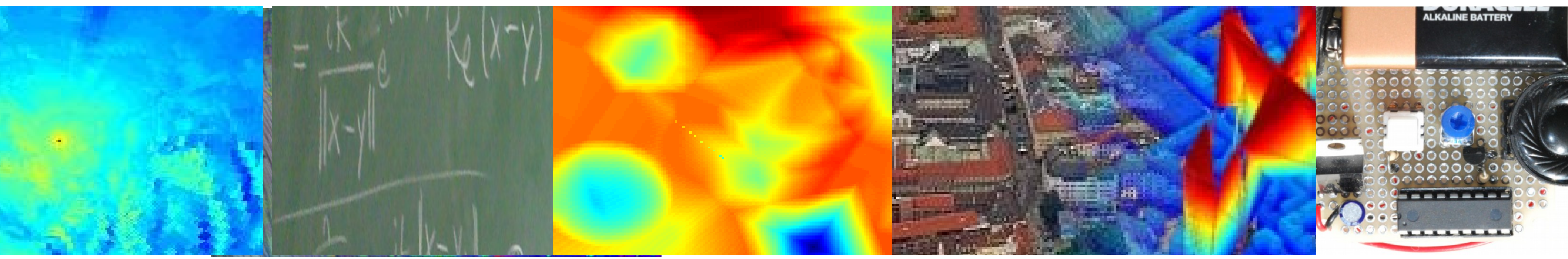
Computation *of the* Godement Resolution *for* Sheaves over Finite Spaces



Michael Robinson

Acknowledgements

Partial funding: Reza Ghanadan (DARPA/DSO)

Collaborators:

- Cliff Joslyn & Emilie Hogan (PNNL)
- Chris Capraro (SRC)

Current students:

Ocean winds

- Tara Shreve
- Dave D'Auria

Sonar processing

- Jen Dumiak
- Leo Brody
- Brian DiZio

Networks

- Danielle Beard
- Samara Fantie
- Eyerusalem Abebe
- Jimmy Palladino

DARPA SIMPLEX

- Philip Coyle
- Janelle Henrich

Pacific Northwest NATIONAL LABORATORY

Proudly Operated by Battelle Since 1965

Content Acknowledgement

Most of the content of this talk was synthesized from the following sources (in roughly descending order):

- J. Curry, "Sheaves, cosheaves, and applications," Ph.D. dissertation, University of Pennsylvania, 2014.
- A. Shepard, "A cellular description of the derived category of a stratified space," Ph.D. dissertation, Brown University, 1985.
- G. Bredon, *Sheaf Theory*, Springer, 1997.
- S. Gelfand and Y. Manin, *Methods of Homological Algebra*, Springer, 1996.
- M. Kashiwara and P. Schapira, *Categories and Sheaves*, Springer, 2005.
- M. Robinson, *Topological Signal Processing*, Springer, 2014.
- http://mathoverflow.net/questions/1151/sheaf-cohomology-and-injective-resolutions

DARPA Tutorial on Sheaves in Data Analytics

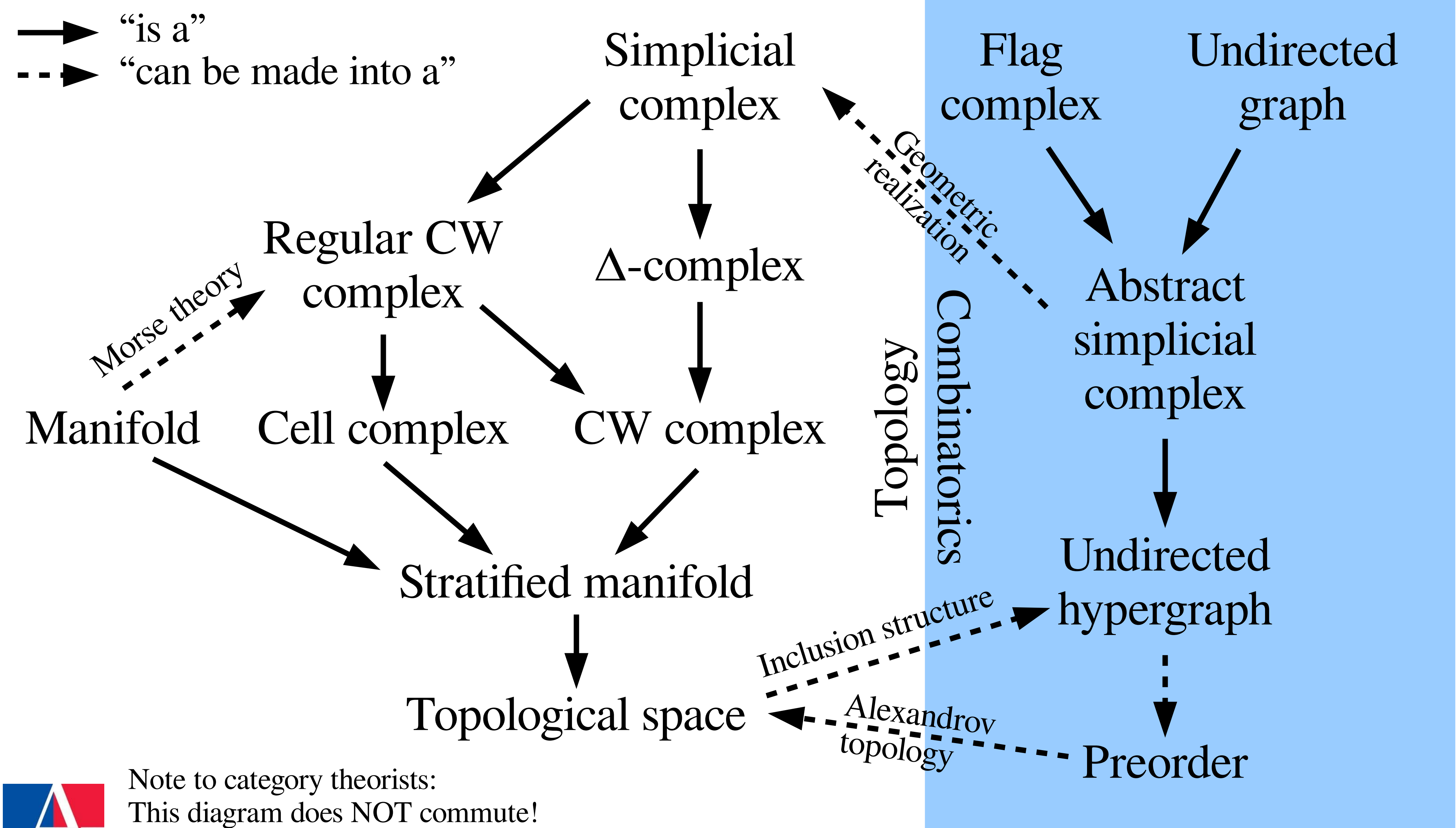
- August 25 and 26, 2015 (already occurred)
- American University, Washington, DC and online
- Websites:

  http://drmichaelrobinson.net/sheaftutorial/index.html

  http://www.american.edu/cas/darpasheaves/index.cfm

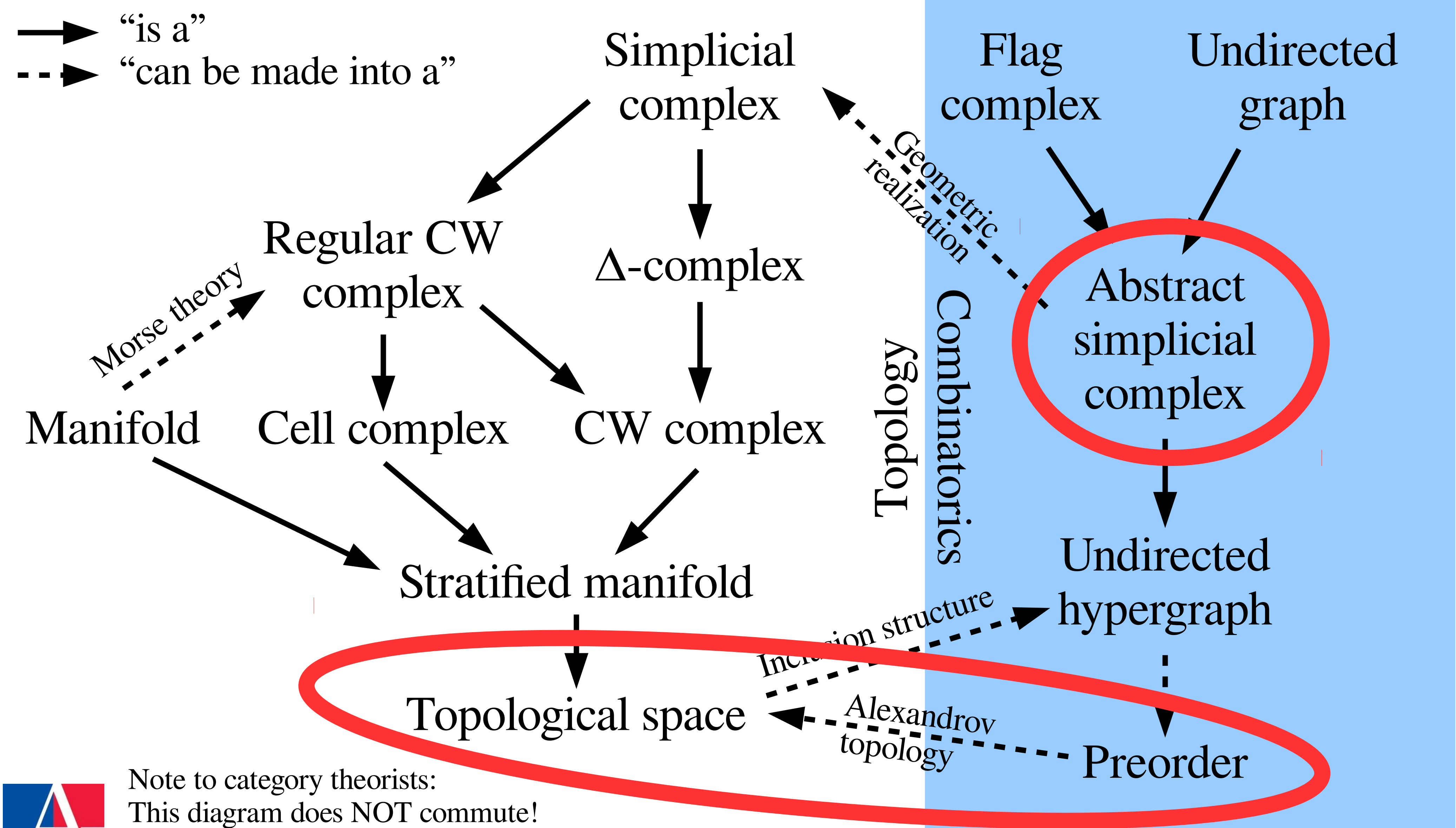
What is a finite space?

Some (mostly) topological spaces



Note to category theorists:
This diagram does NOT commute!

Some (mostly) topological spaces



Note to category theorists:
This diagram does NOT commute!

Topology

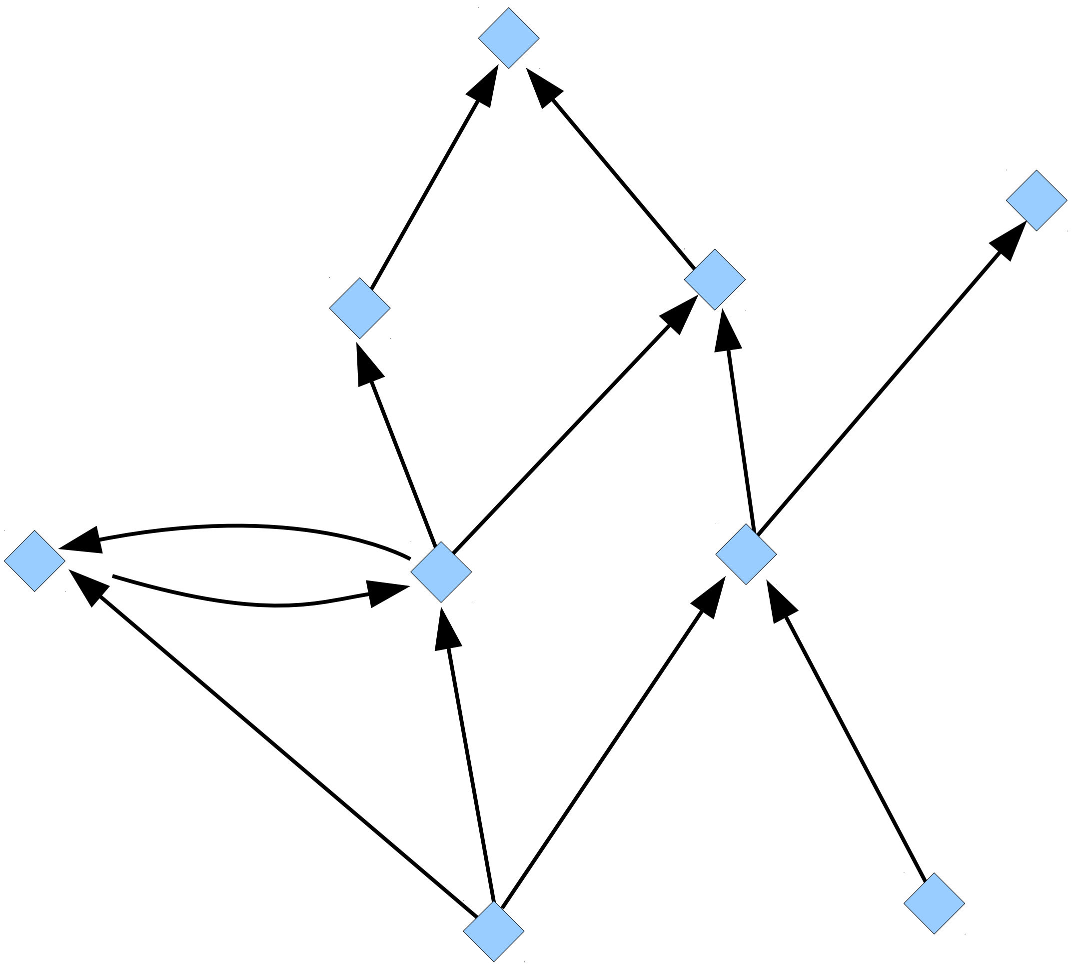
If X is a set, then a *topology* T on X is a collection of subsets of X satisfying four axioms:

1. The empty set is in T
2. X is in T
3. If U is in T and V is in T then $U \cap V$ is in T
4. All unions of elements of T are in T

The elements of T are called the *open sets* for the topology

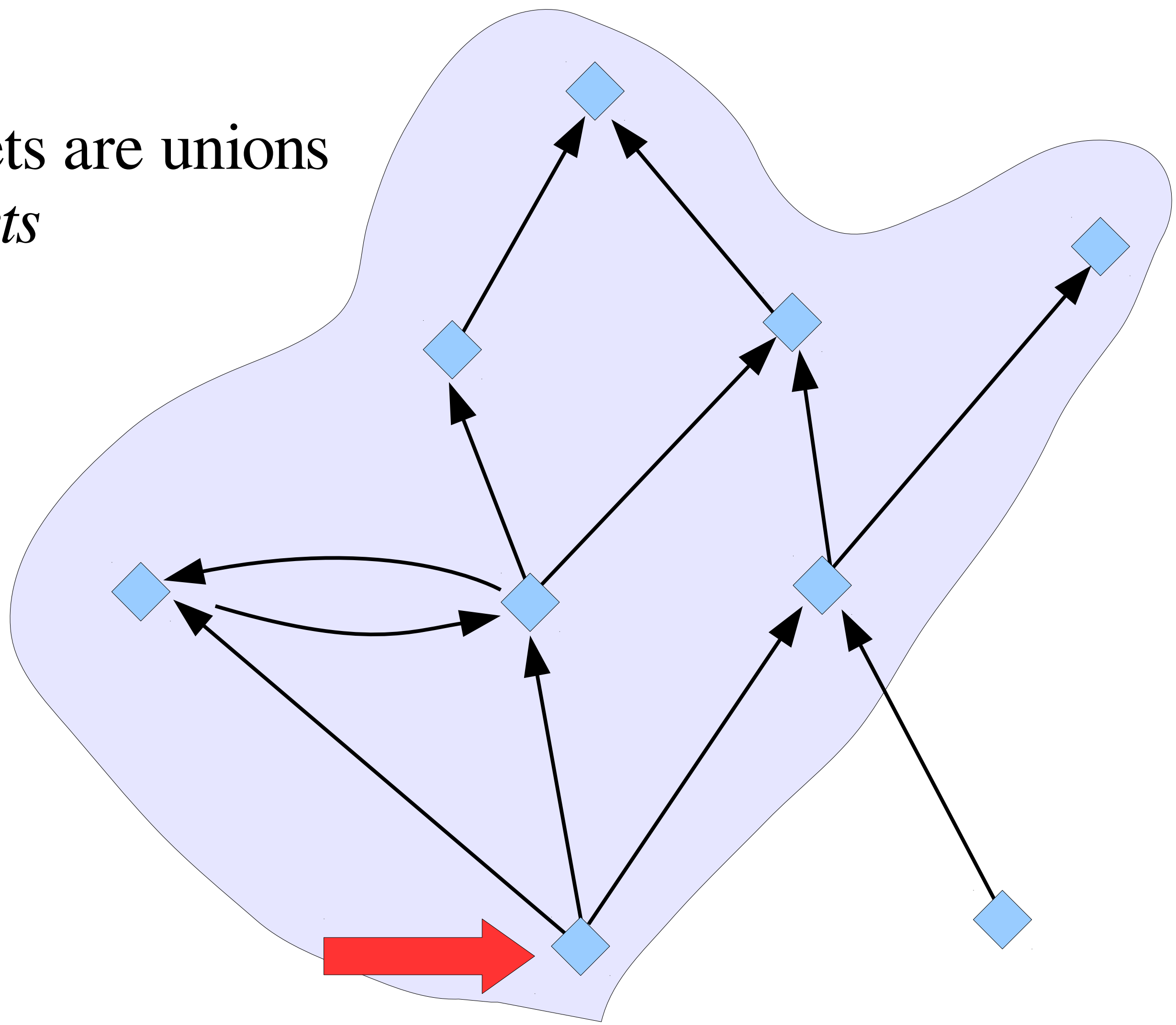
The pair (X,T) is called a *topological space*

Topologizing a preorder



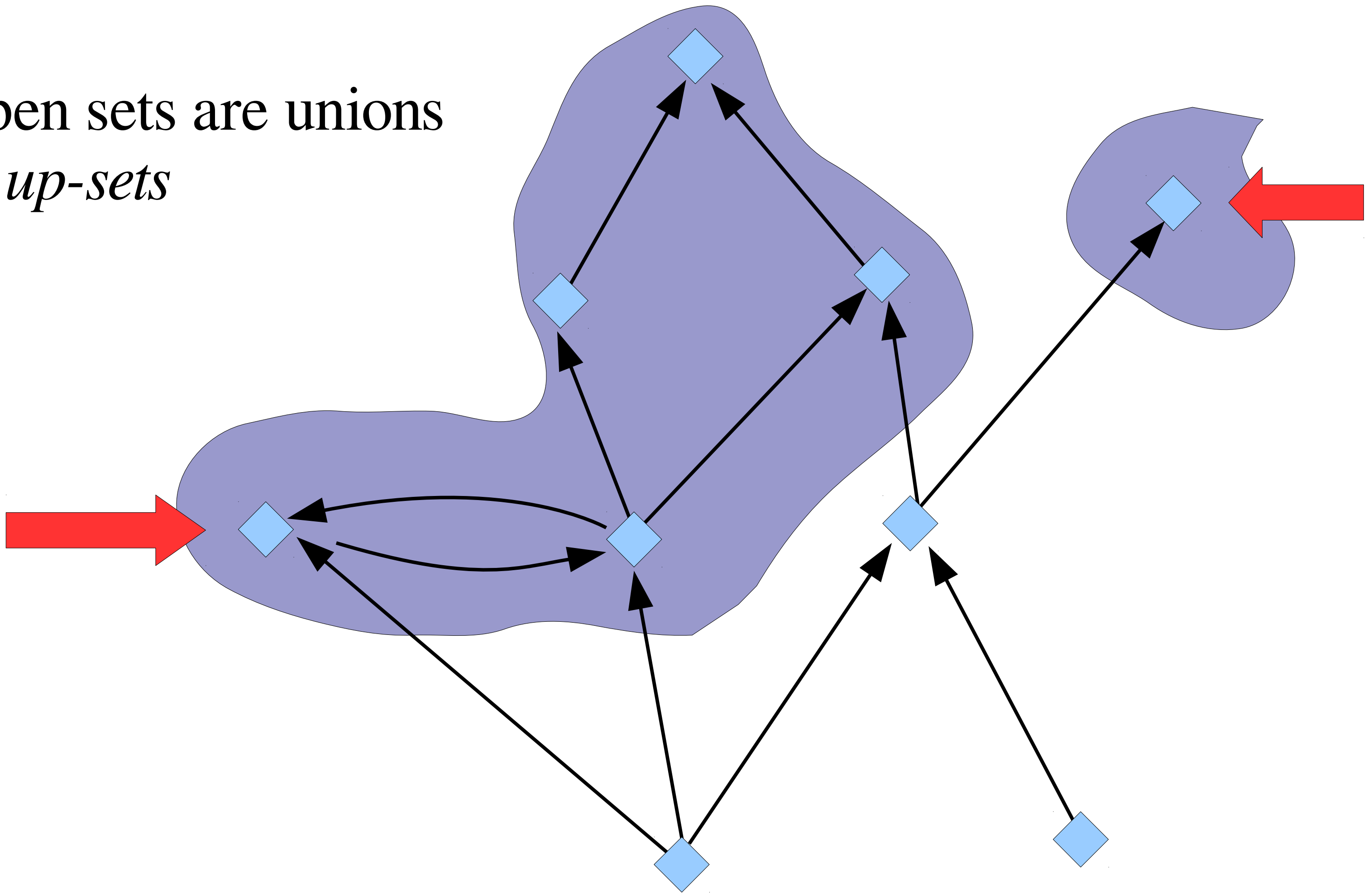
Topologizing a preorder

Open sets are unions of *up-sets*



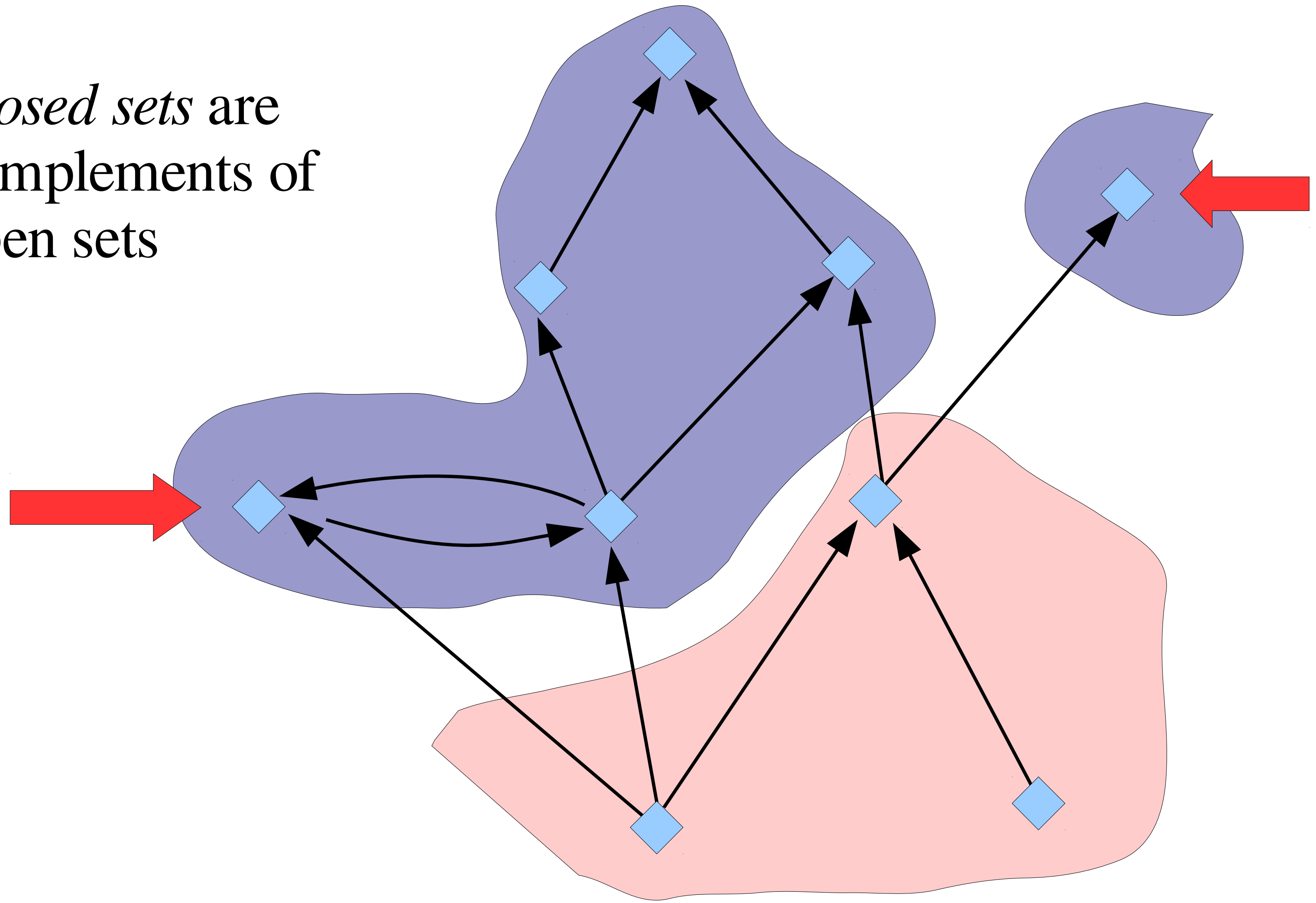
Topologizing a preorder

Open sets are unions of *up-sets*



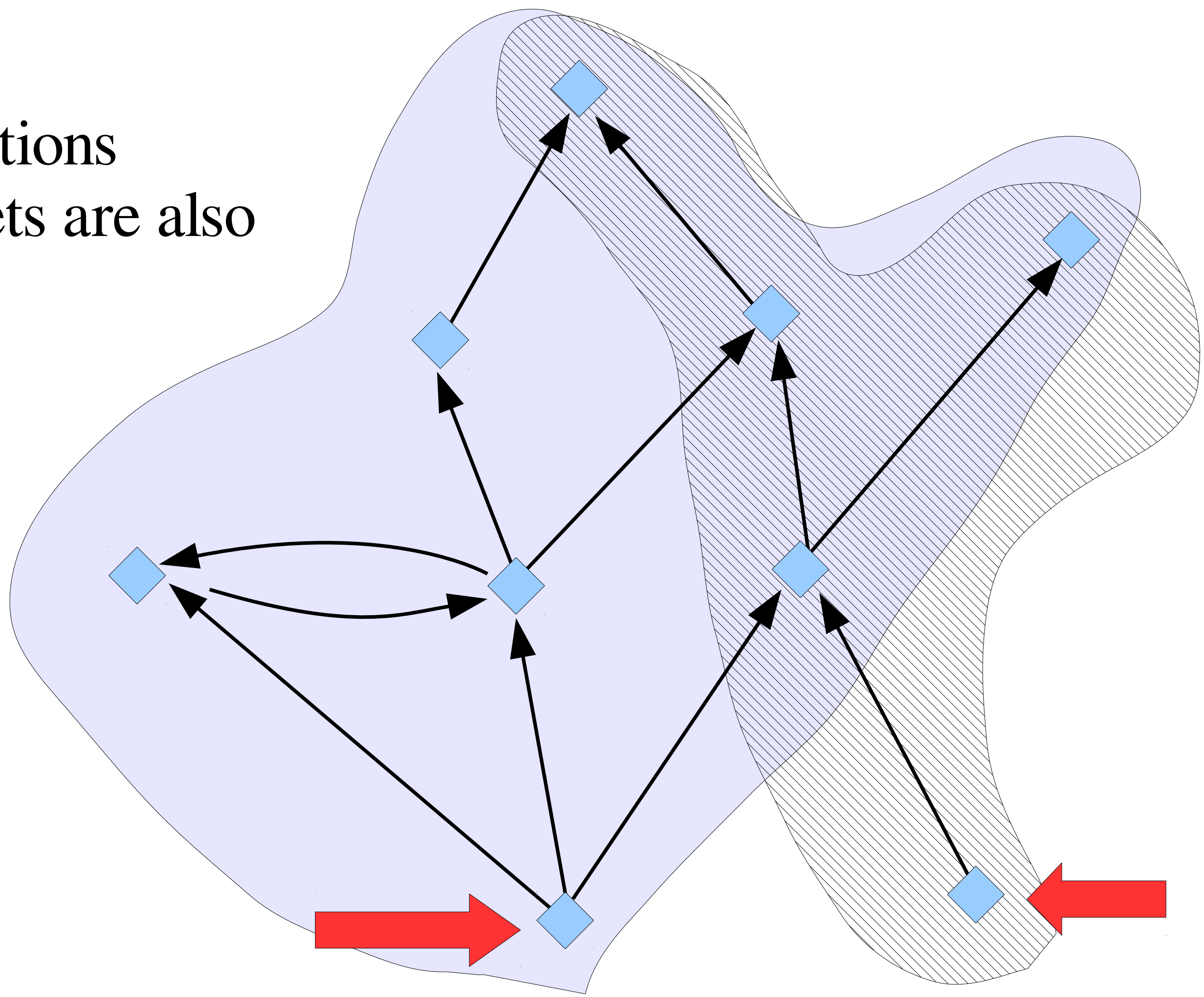
Topologizing a preorder

Closed sets are complements of open sets



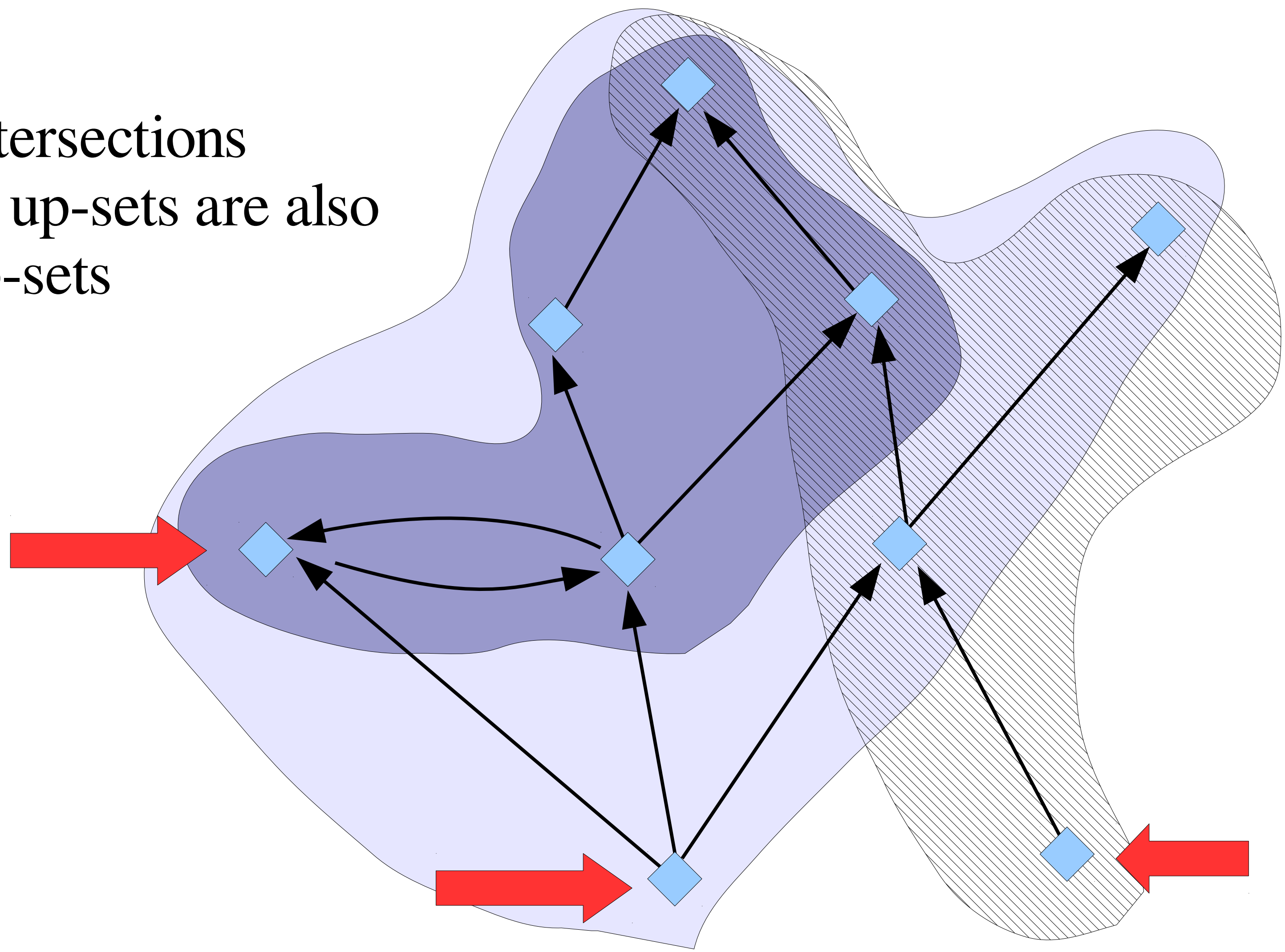
Topologizing a preorder

Intersections of up-sets are also up-sets



Topologizing a preorder

Intersections of up-sets are also up-sets



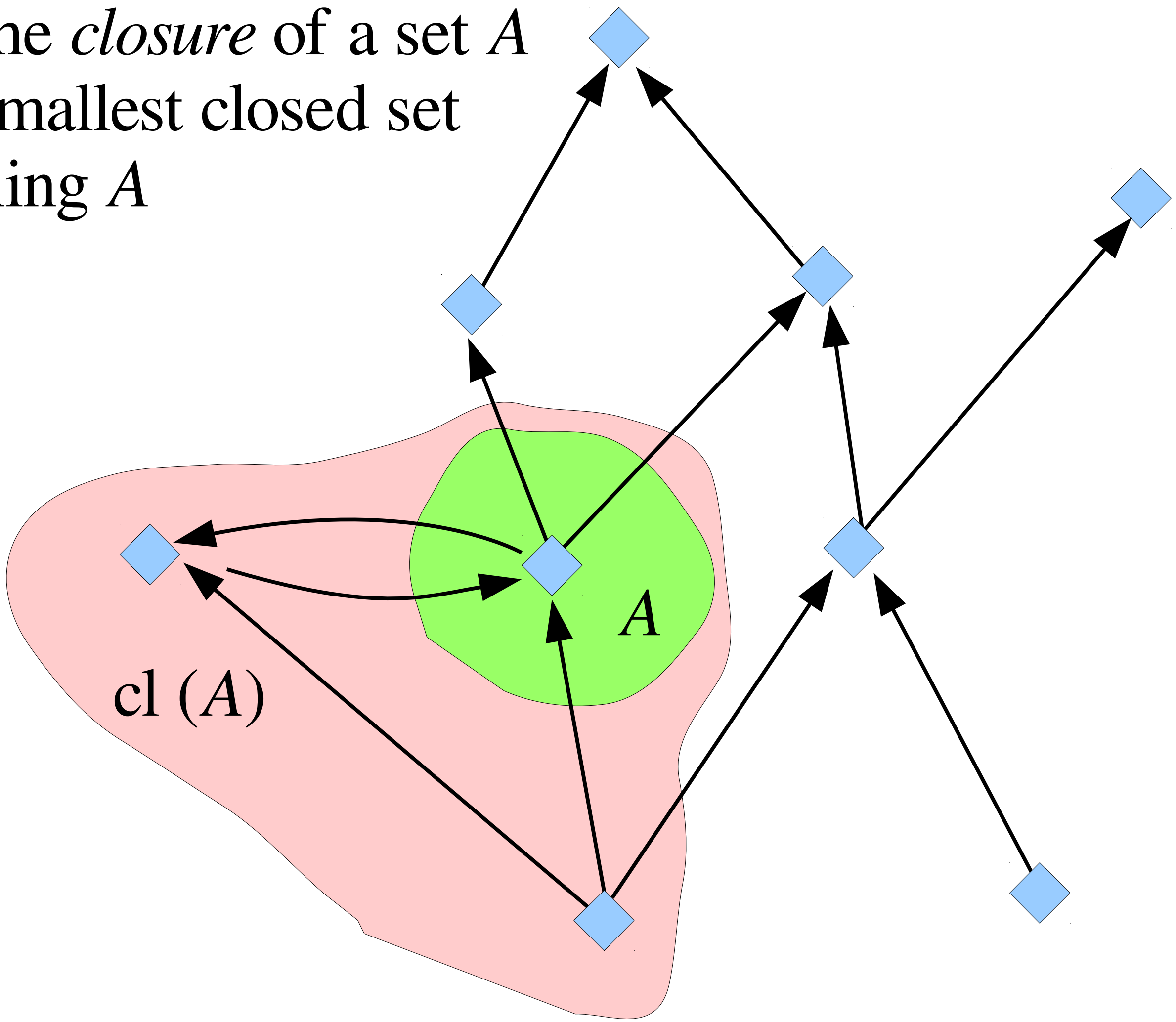
Alexandrov spaces

- A topological space in which **arbitrary** intersections of open sets are still open is called *Alexandrov*
- Proposition: All topological spaces over a finite set are Alexandrov

- All topological spaces have *closures*...
- ...but Alexandrov spaces also have *stars*

Closures

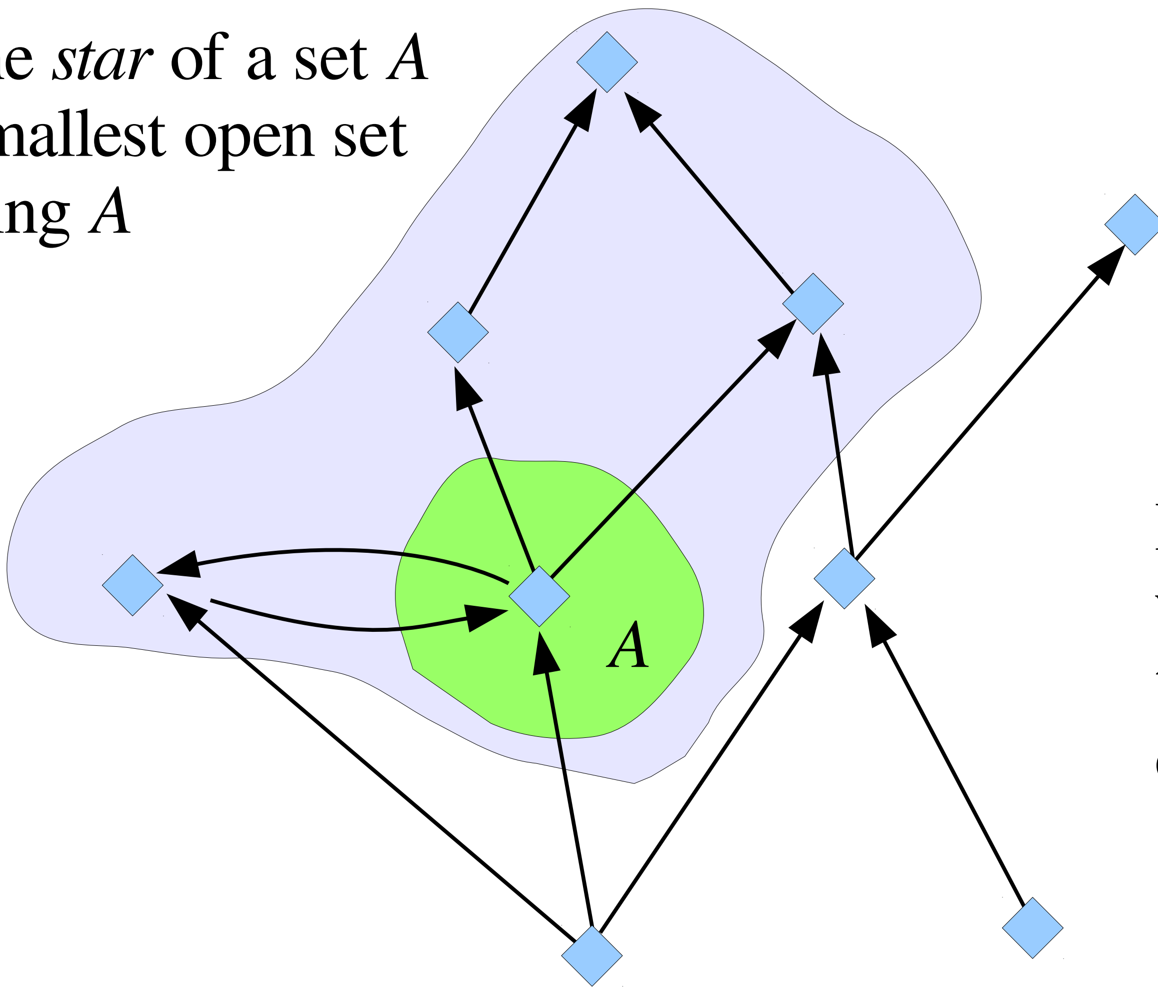
Def: The *closure* of a set A is the smallest closed set containing A



(All topological spaces can form the closure of any subset)

Stars

Def: The *star* of a set A is the smallest open set containing A

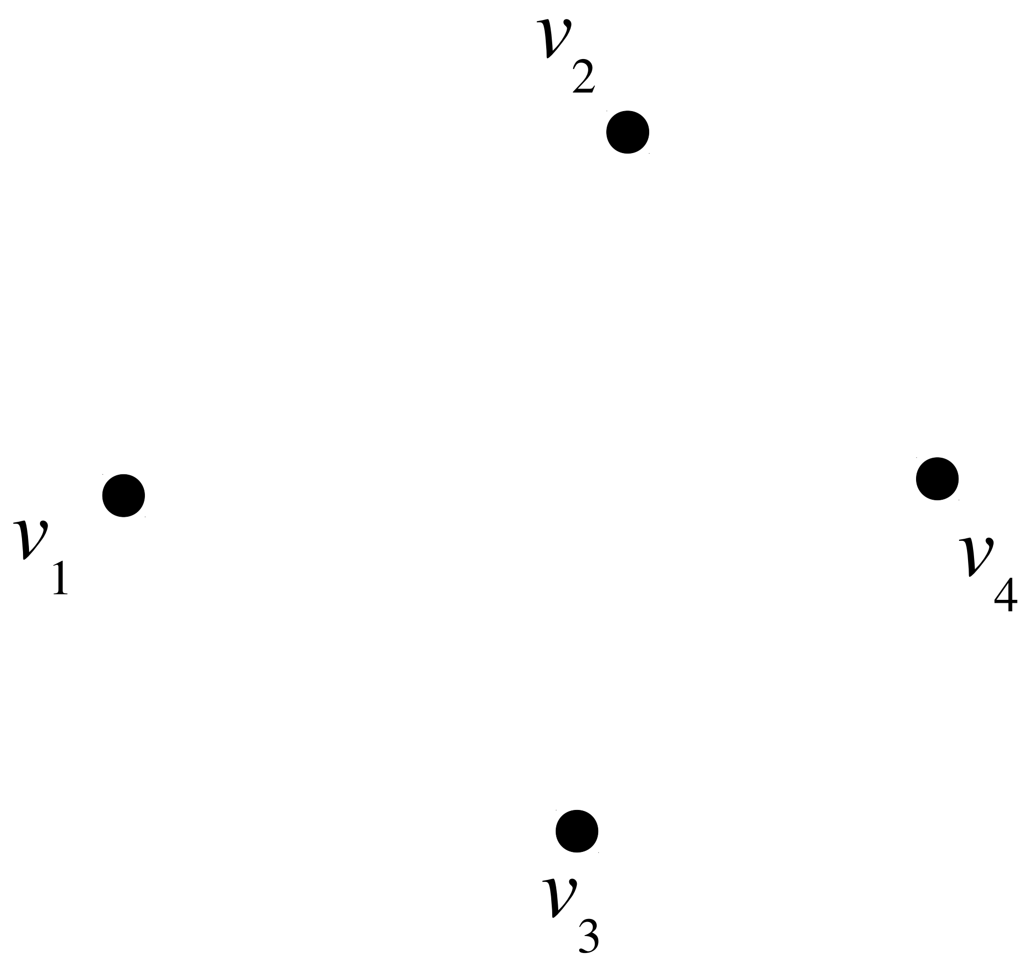


Don't confuse this with the *interior* of A: the **largest** open set contained **within** A

(Some topological spaces cannot form stars)

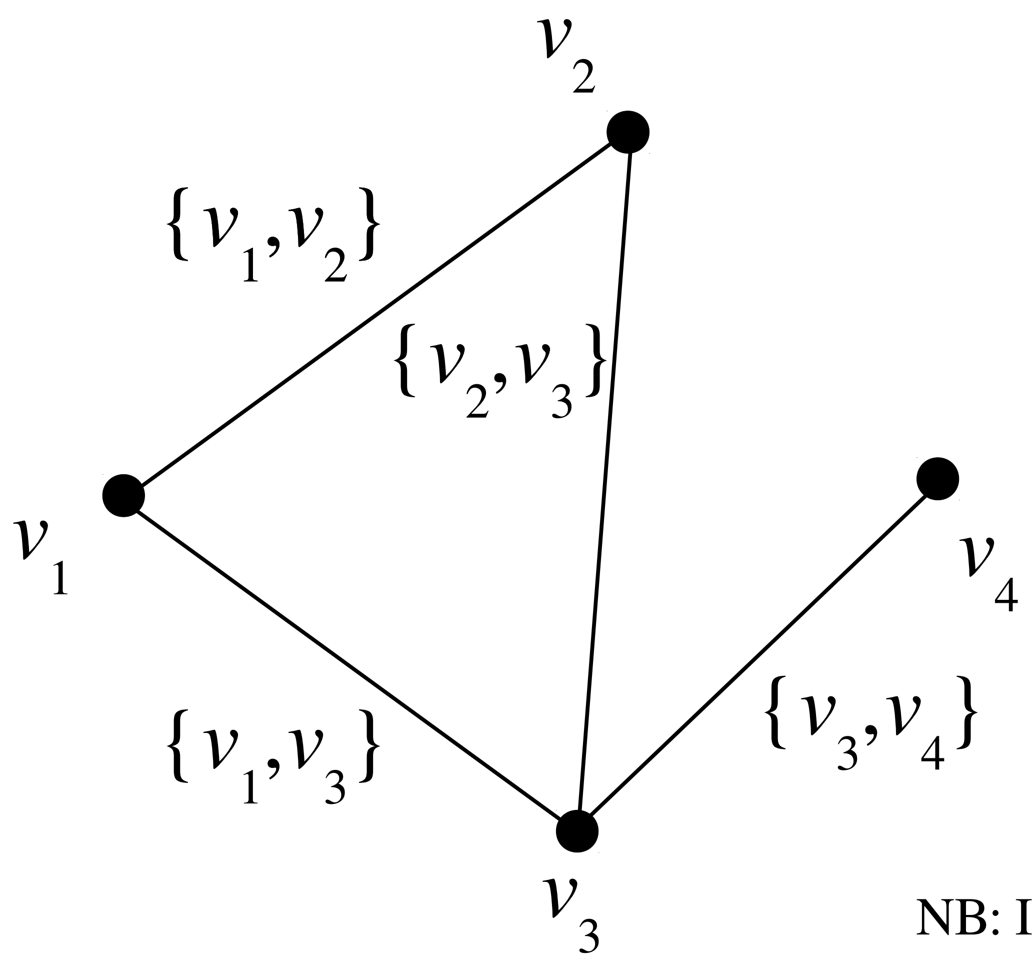
Simplicial complexes

- A simplicial complex is a collection of *vertices* and ...



Simplicial complexes

- … *edges* (pairs of vertices) and ...

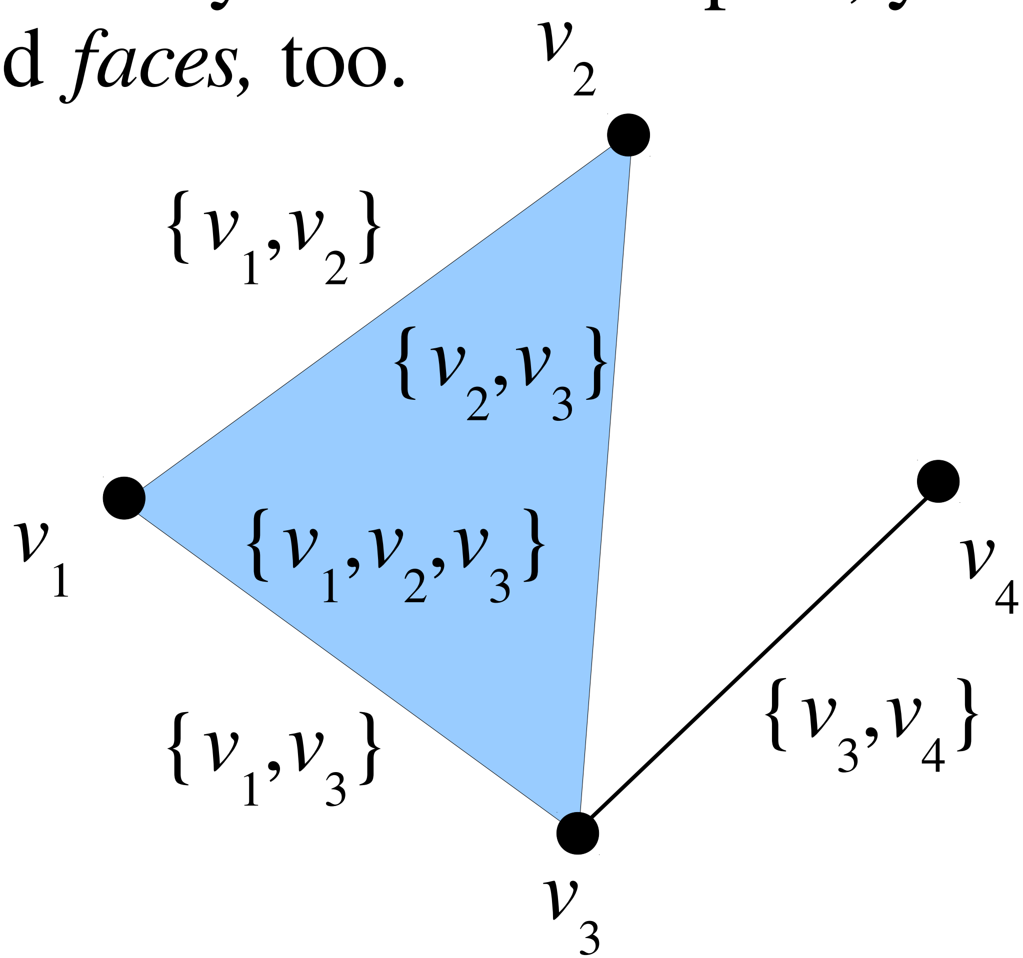


Usually people call this a *graph*; I will too

NB: I got lazy… v_1 should be $\{v_1\}$

Simplicial complexes

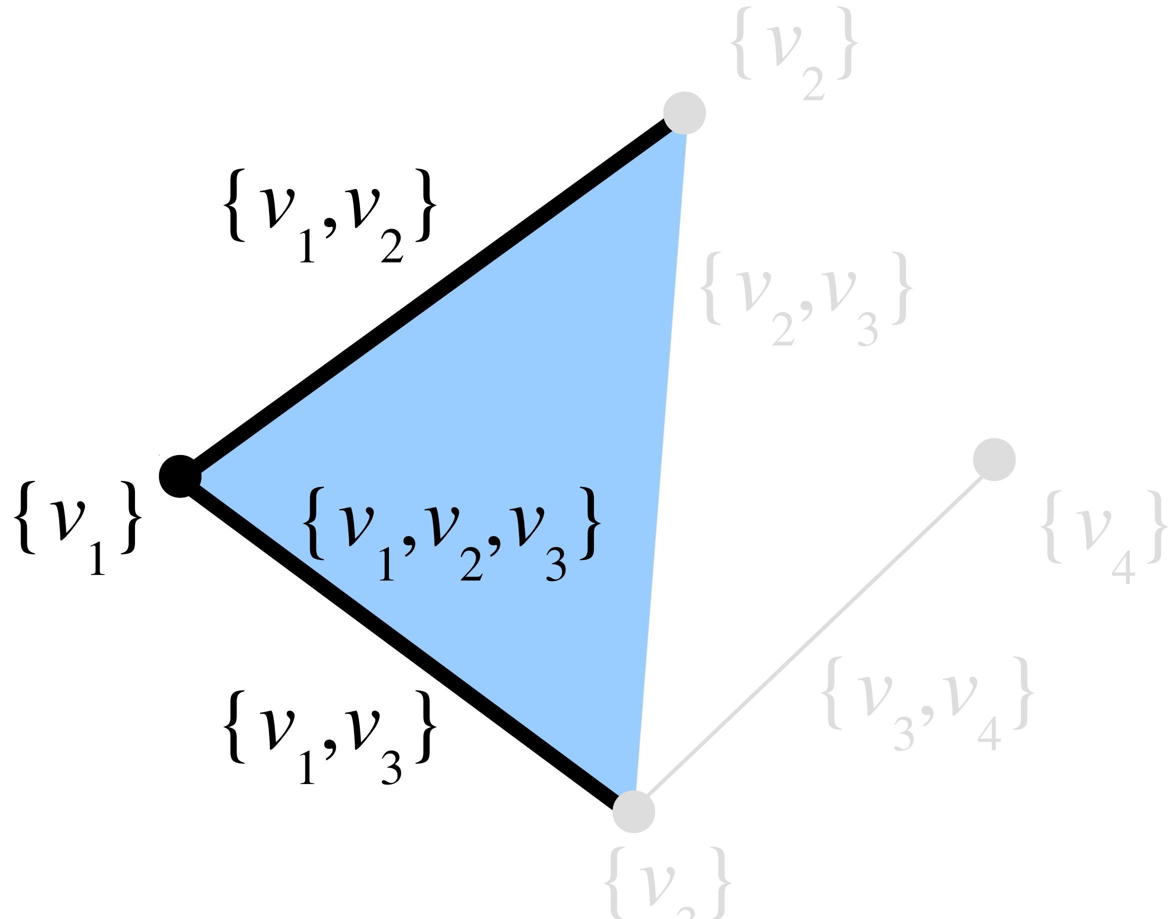
- ... higher dimensional *simplices* (tuples of vertices)
- Whenever you have a simplex, you have all subsets, called *faces*, too.



Stars over simplices

- The *star* over a simplex is that simplex along with all higher dimensional ones containing it

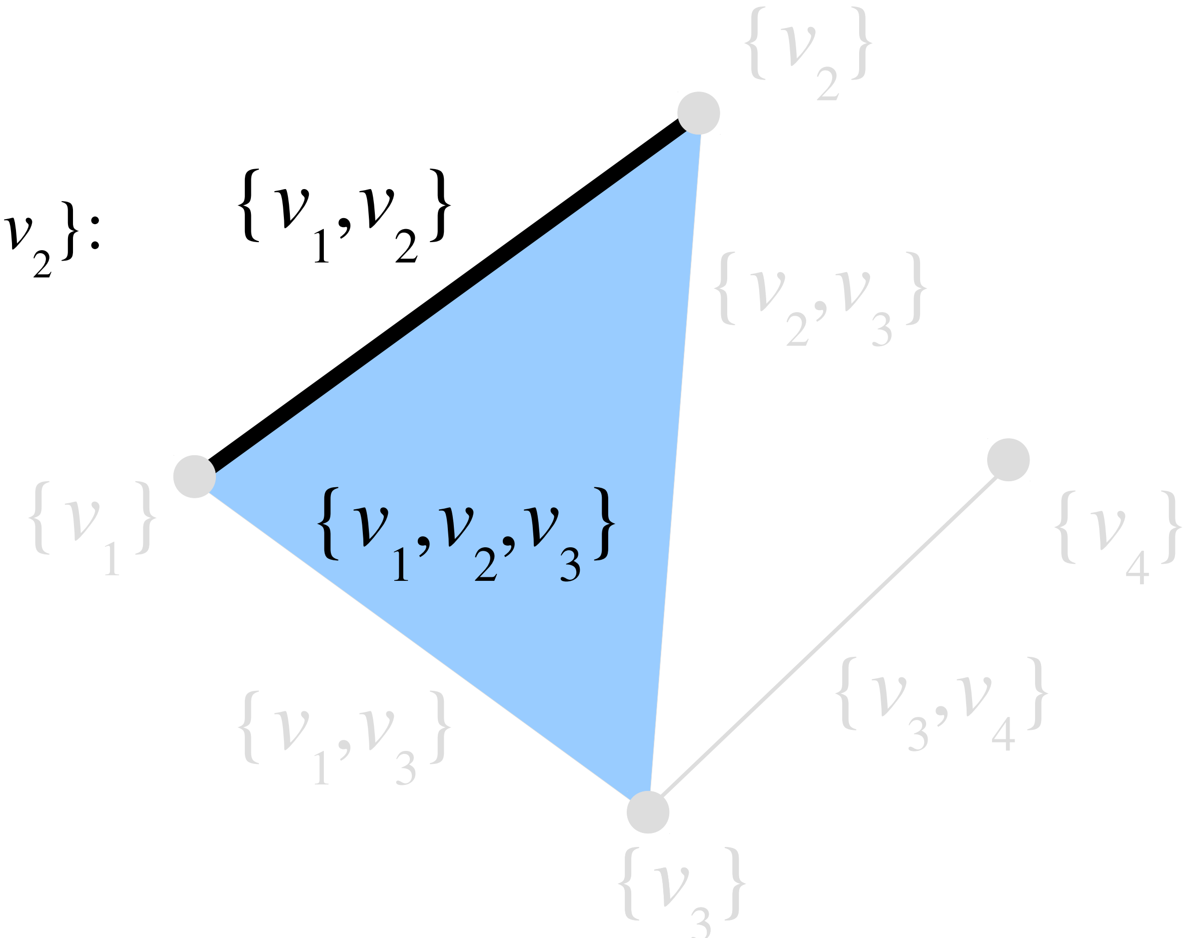
Star over v_1:



Stars over simplices

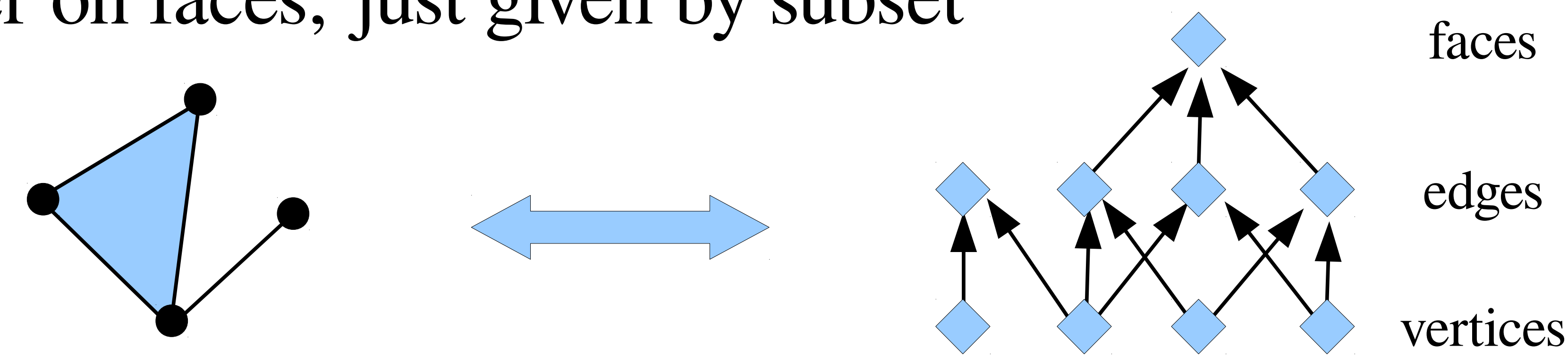
- The *star* over a simplex is that simplex along with all higher dimensional ones containing it

Star over $\{v_1,v_2\}$:



Stars over simplices

- Abstract simplicial complexes have a canonical partial order on faces, just given by subset



Simplicial complex

Partial order

- <u>Proposition</u>: The collection of all possible stars and all possible unions of stars forms a topology for the set of simplices

…. and so the word *star* has the same meaning in both contexts

What is a sheaf?

A *sheaf* of ________________ on a _________________

(data type) (topological space)

Sheaves: a definition

A sheaf on a topological space X consists of

A contravariant functor F from **Open**(X) to some subcategory of **Vec**; this is a "sheaf of vector spaces"

- $F(U)$ for open U is called the *space of sections* over U
- Each inclusion map $U \subset V$ is sent to a linear *restriction map* $F(V) \to F(U)$.
- Given a point $p \in X$, the colimit of $F(U_\alpha)$, for all U_α satisfying $p \in U_\alpha$ is called the *stalk* at p. It's a generalization of the germ of a smooth function

And a gluing rule...

Sheaves: a definition

A sheaf on a topological space X consists of

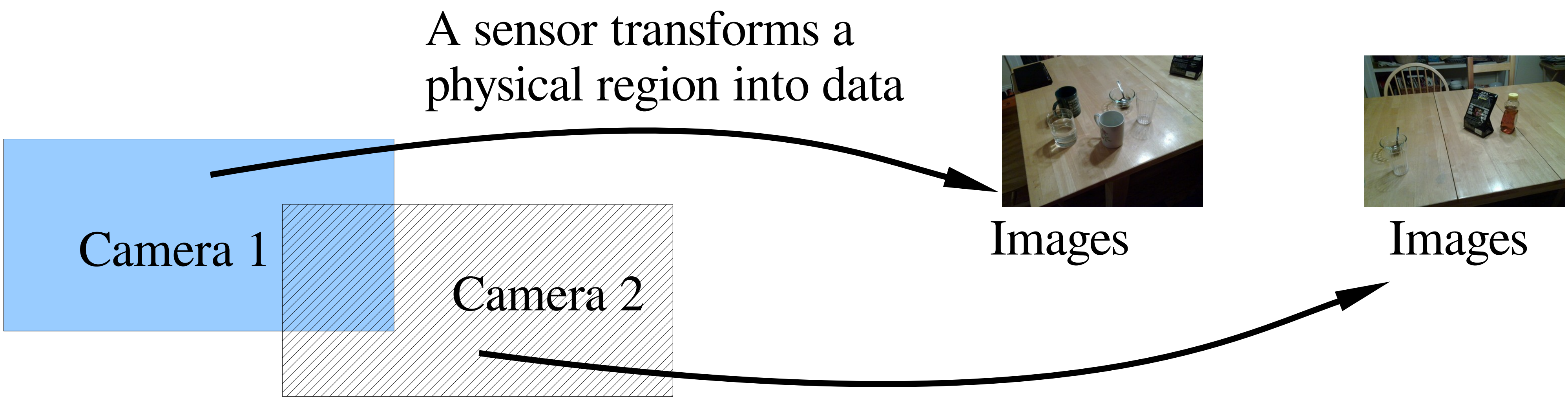
A contravariant functor F from **Open**(X) to so… subcategory of **Vec**; this is a "sheaf of …ces"

- $F(U)$ for open U is called the *spa*… over U
- Each inclusion map $U \subseteq V$ … linear *restriction map* $F(V) \rightarrow F(U)$.
- Given a … the colimit of $F(U_\alpha)$, for all U_α satisfying … called the *stalk* at *p*. It's a generalization of the germ of a smooth function

And a gluing rule...

Sheaves assign local data to open sets

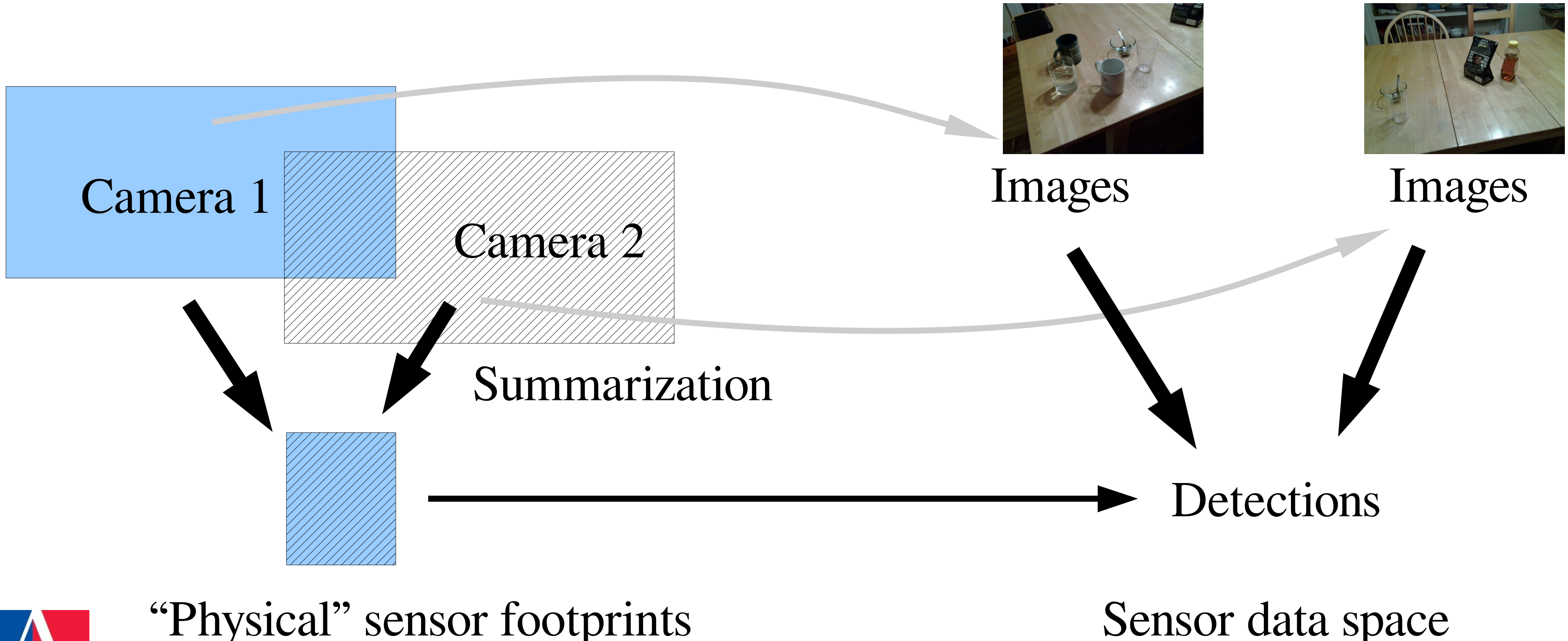
Heterogeneous fusion among homogeneous sensors



"Physical" sensor footprints

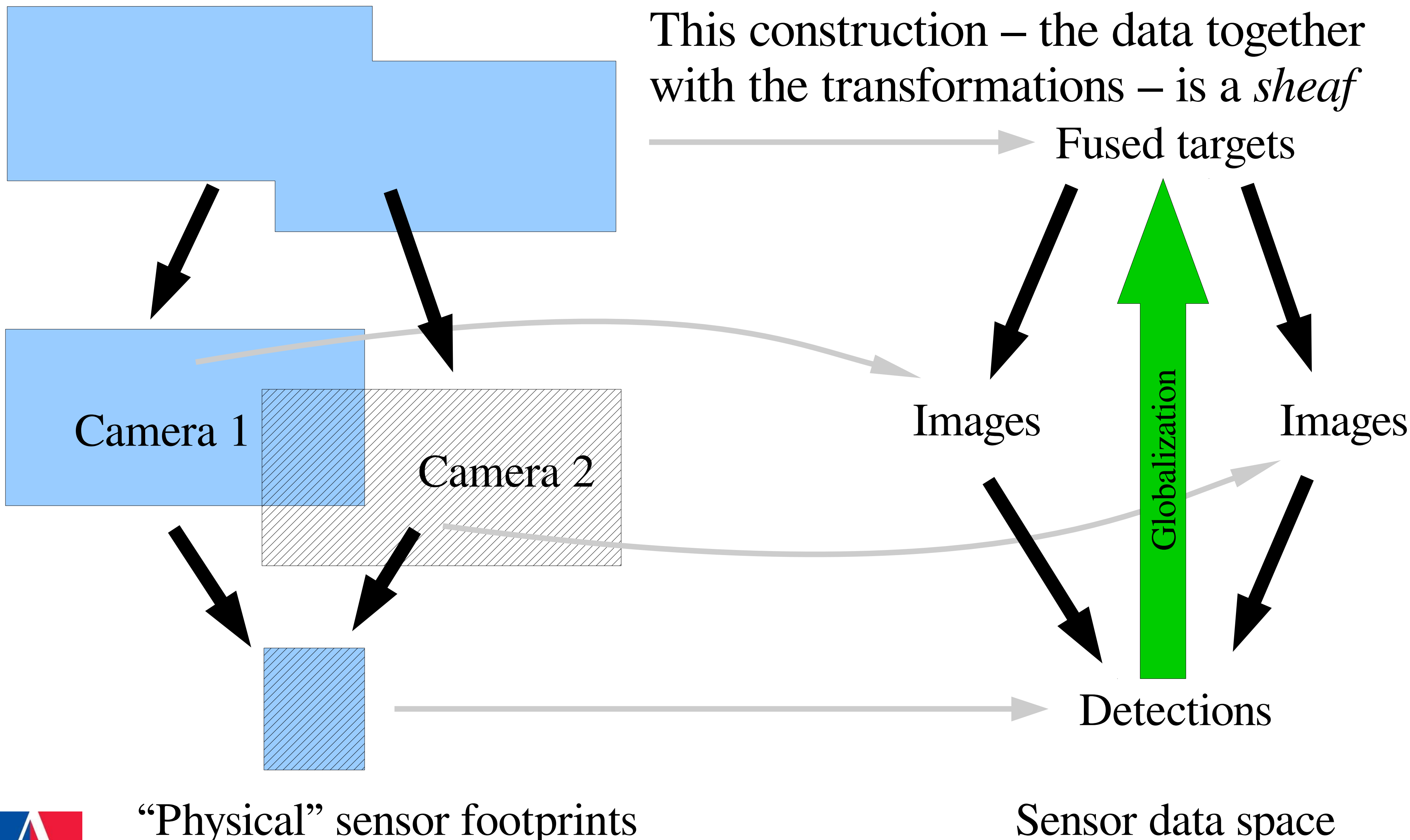
Sensor data space

Heterogeneous fusion among homogeneous sensors

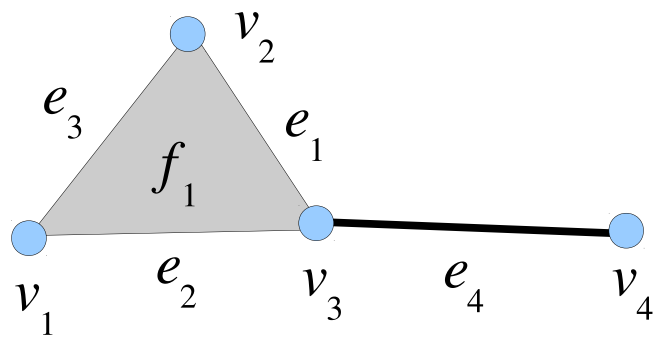


Heterogeneous fusion among homogeneous sensors

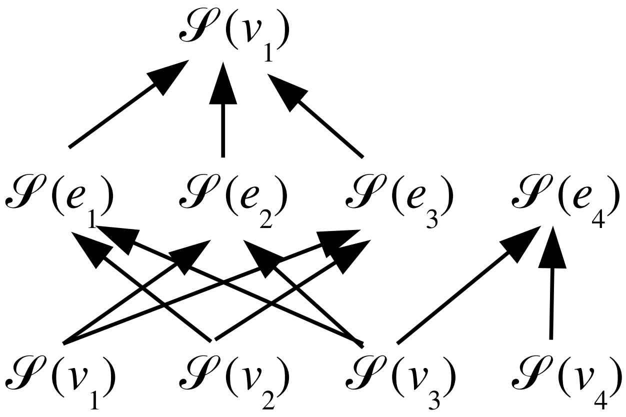
This construction – the data together with the transformations – is a *sheaf*



Extending to collections of faces

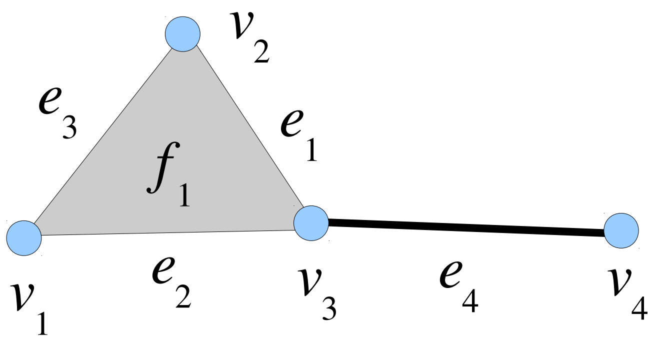


Simplicial complex

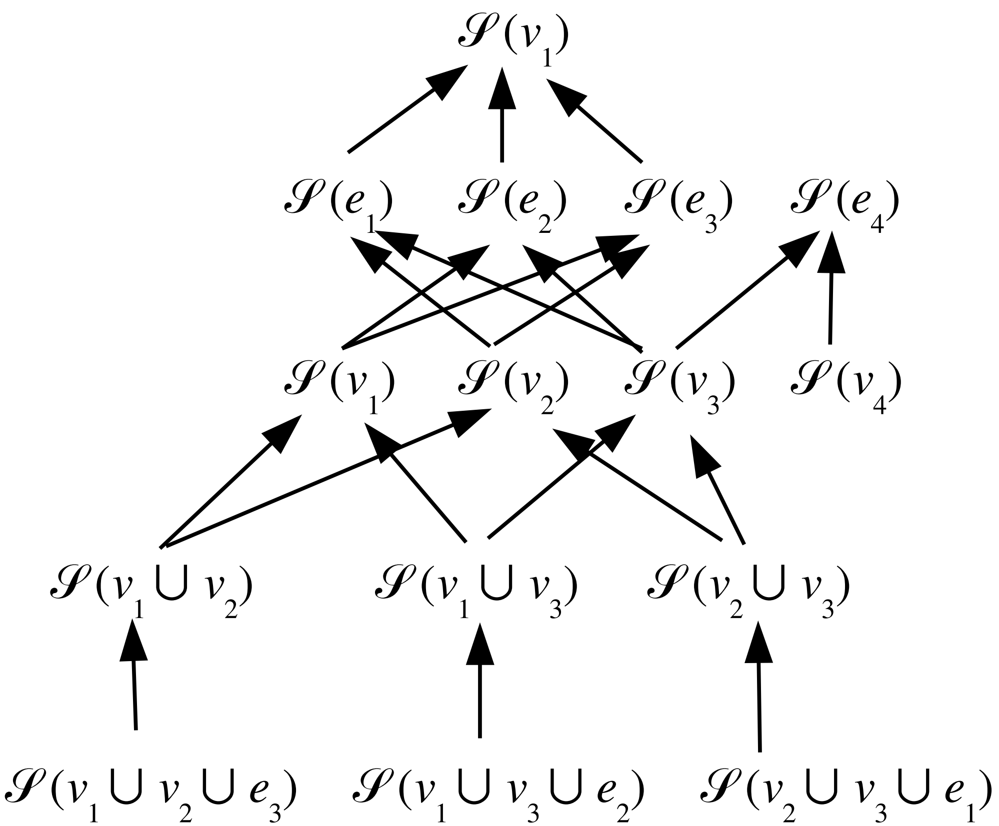


Sheaf assigns data spaces to stars over each simplex

Extending to collections of faces



The data spaces sew together by the gluing rule just described

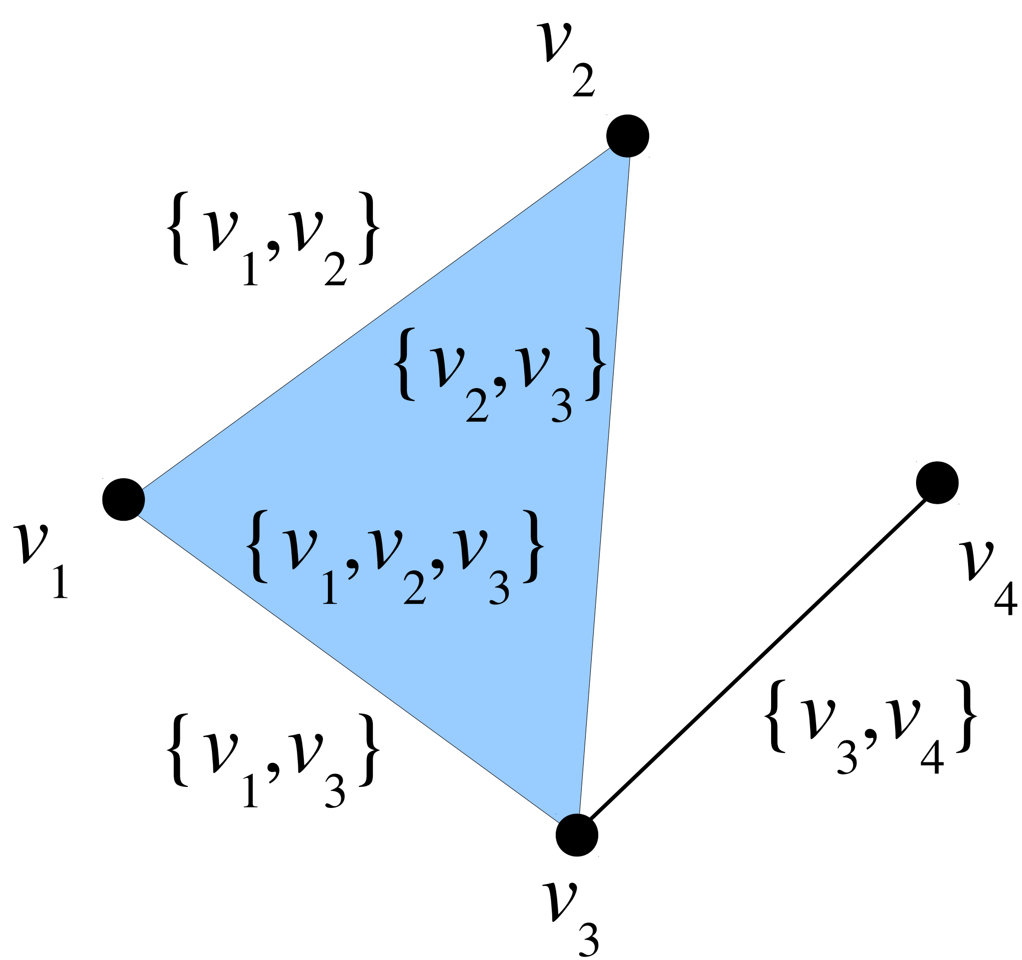


(And many more not shown)

Note: this process is canonical, so we merely need to specify data on a base for the topology and let gluing fill out the rest!

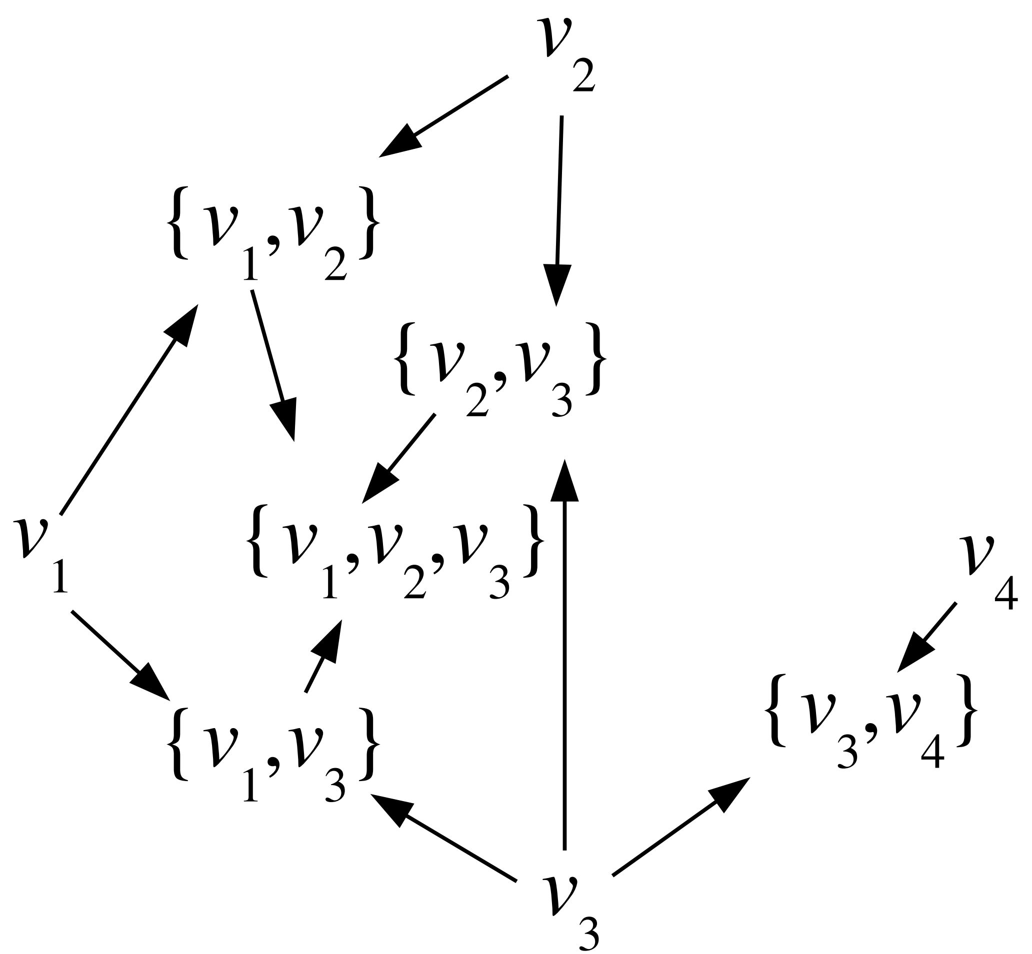
Simplicial complexes

- ... higher dimensional *simplices* (tuples of vertices)



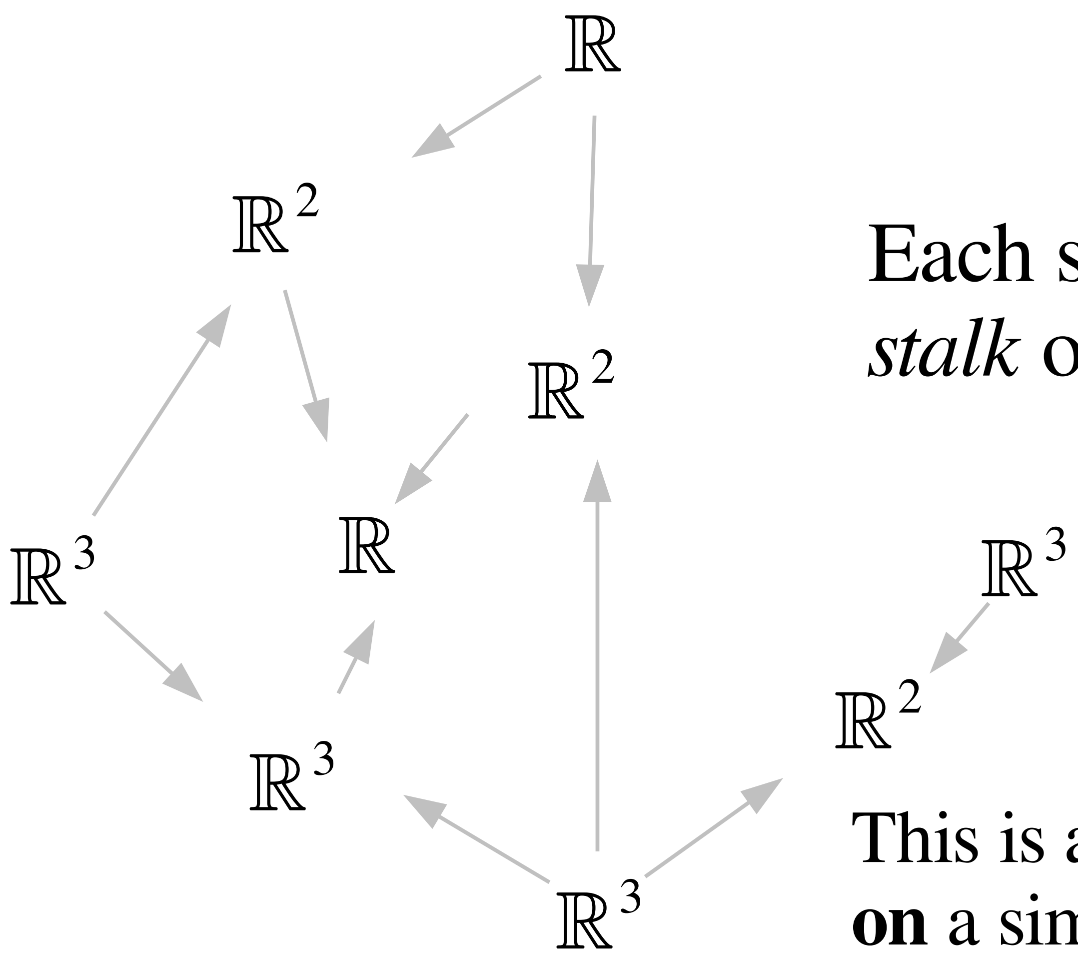
Simplicial complexes

- The *attachment diagram* shows how simplices fit together



A *sheaf* is ...

- A set assigned to each simplex and ...

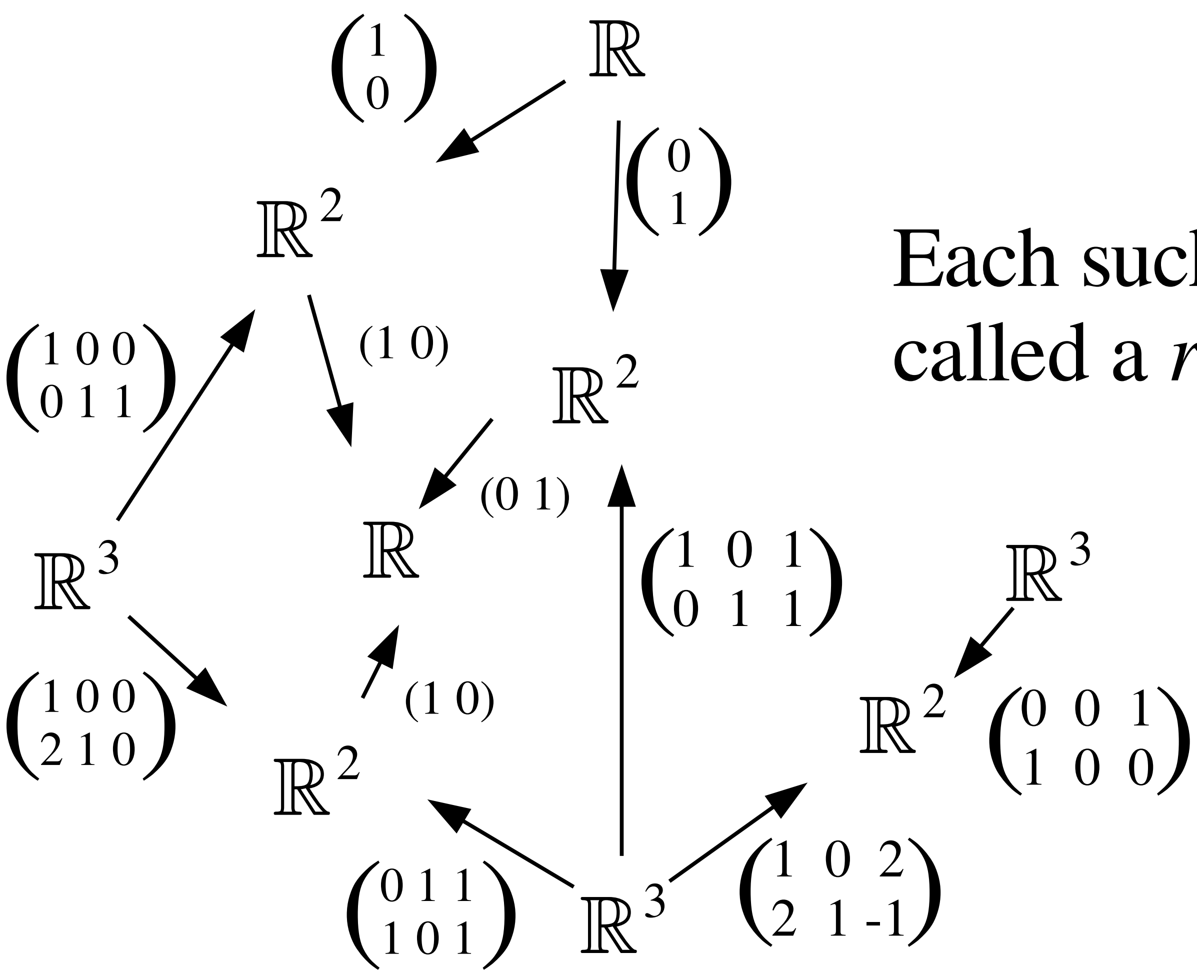


Each such set is called the *stalk* over its simplex

This is a sheaf **of** vector spaces **on** a simplicial complex

A *sheaf* is ...

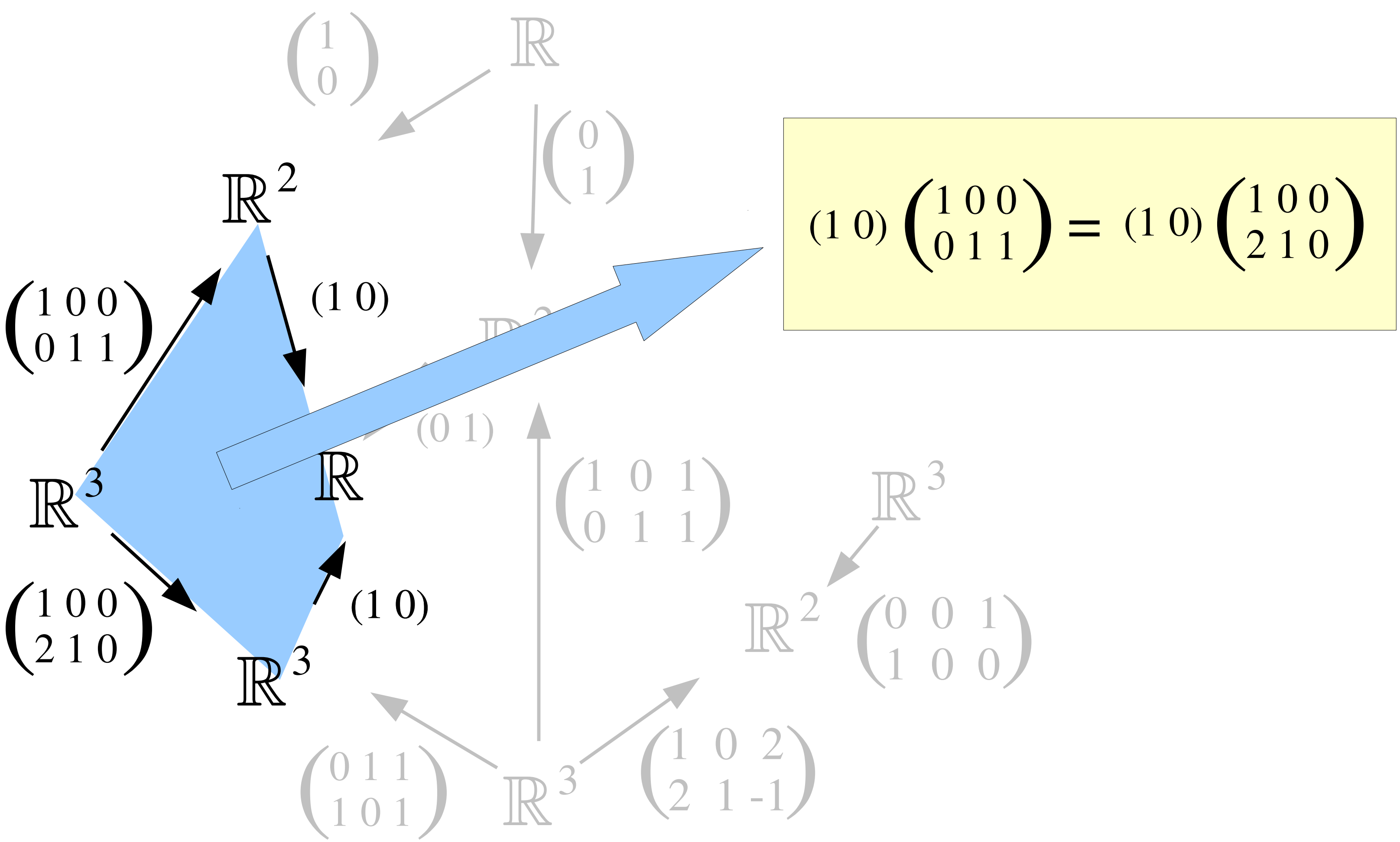
- … a function assigned to each simplex inclusion



Each such function is called a *restriction*

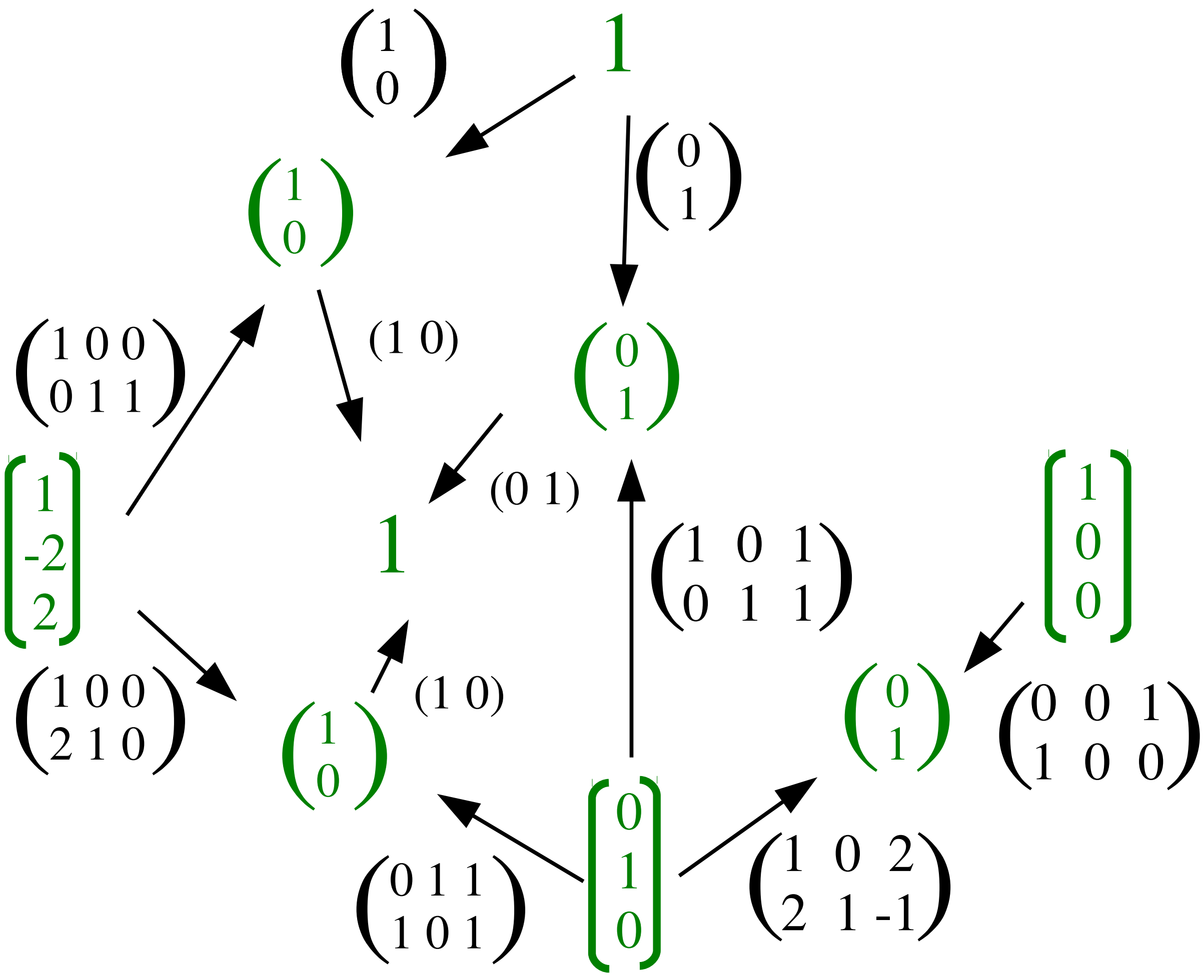
A *sheaf* is ...

- … so the diagram commutes.



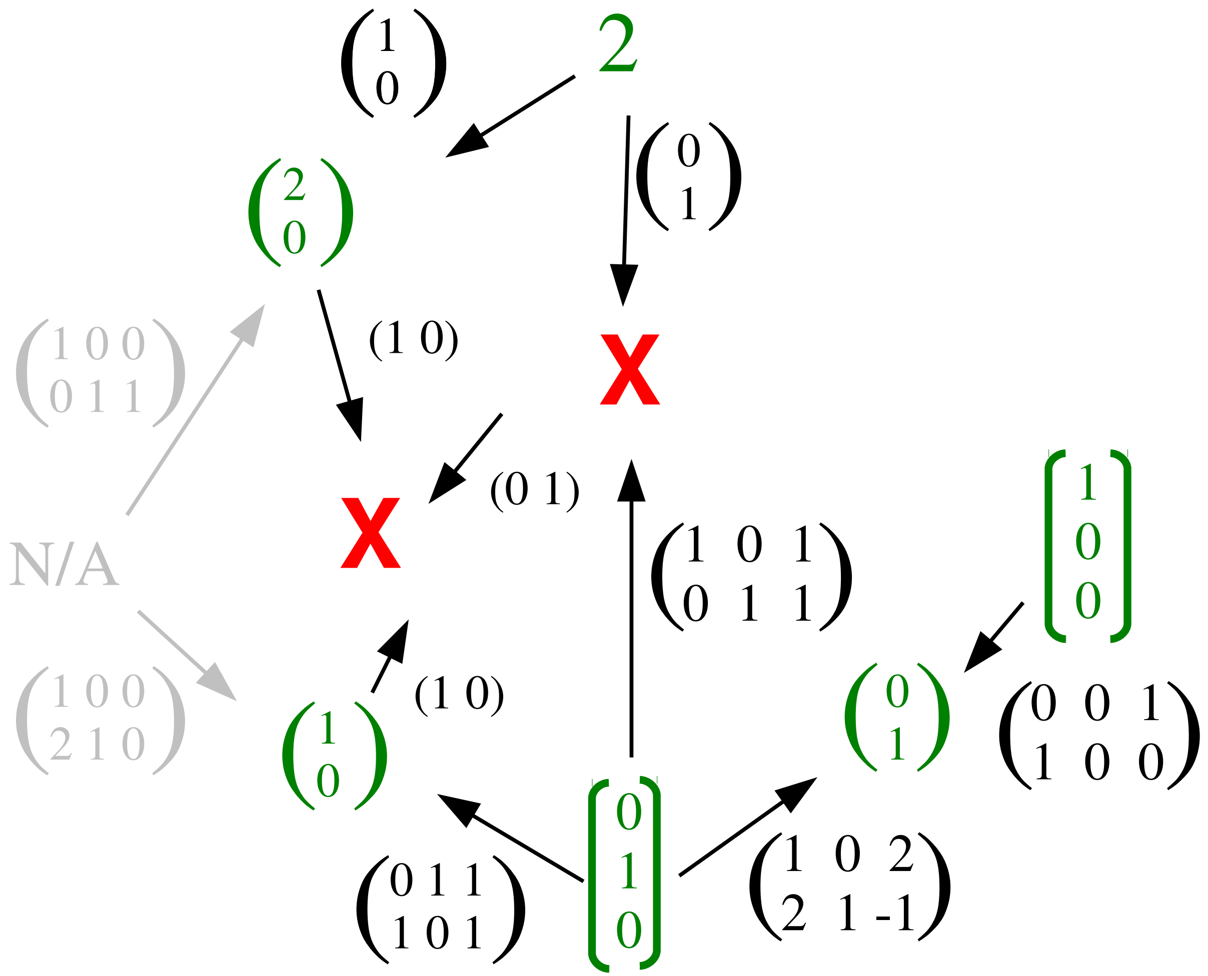
A *global section* is ...

- An assignment of values from each of the stalks that is consistent with the restrictions



Some sections are only local

- They might not be defined on all simplices or disagree with restrictions



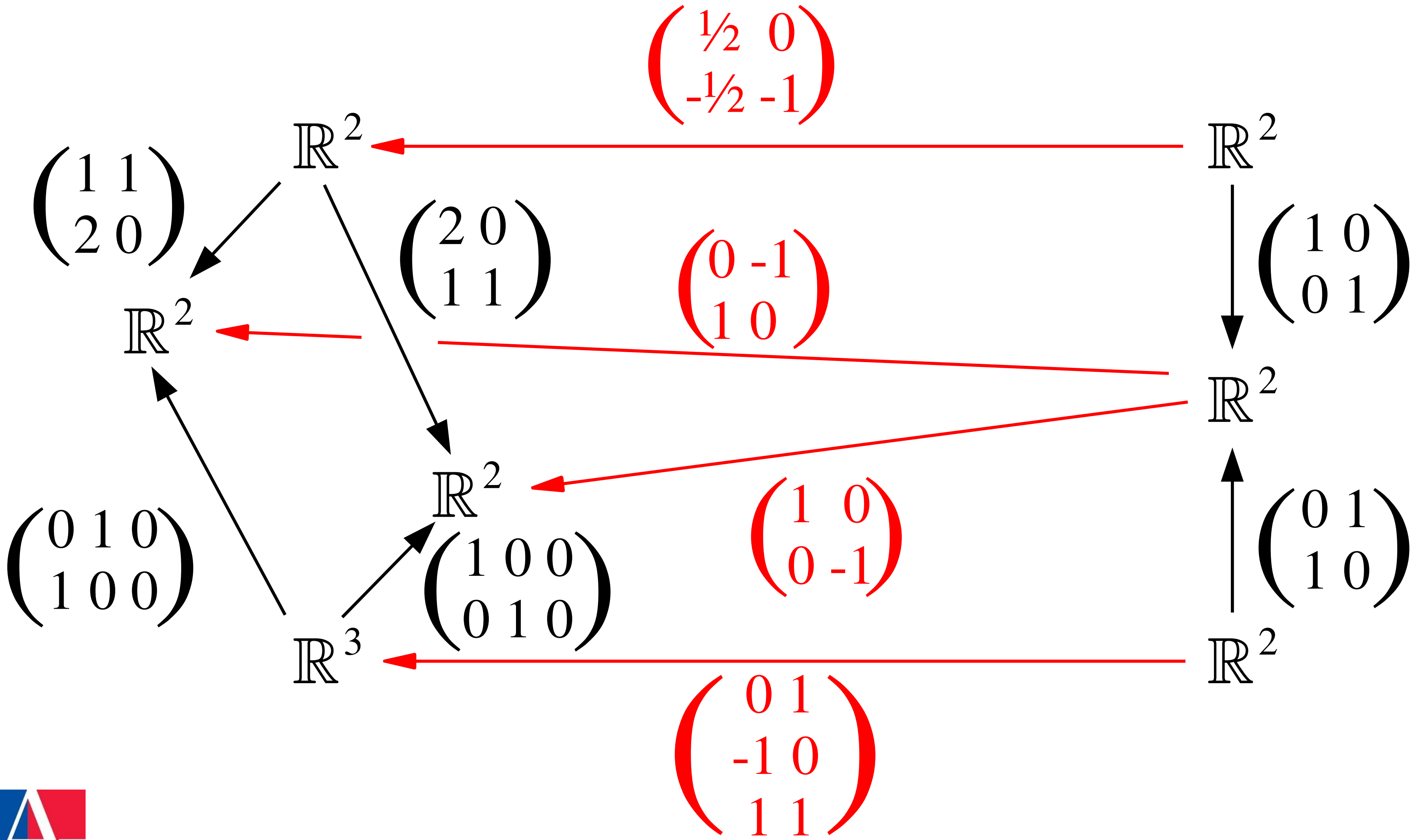
A *sheaf morphism* ...

- … takes data in the stalks of two sheaves …

$$
\begin{tikzcd}
 & \mathbb{R}^2 \arrow[dl, "\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}"'] \arrow[dr, "\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}"] & \\
\mathbb{R}^2 & & \mathbb{R}^2 \\
 & \mathbb{R}^3 \arrow[ul, "\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}"] \arrow[ur, "\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{R}^2 \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"] \\
\mathbb{R}^2 \\
\mathbb{R}^2 \arrow[u, "\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}"']
\end{tikzcd}
$$

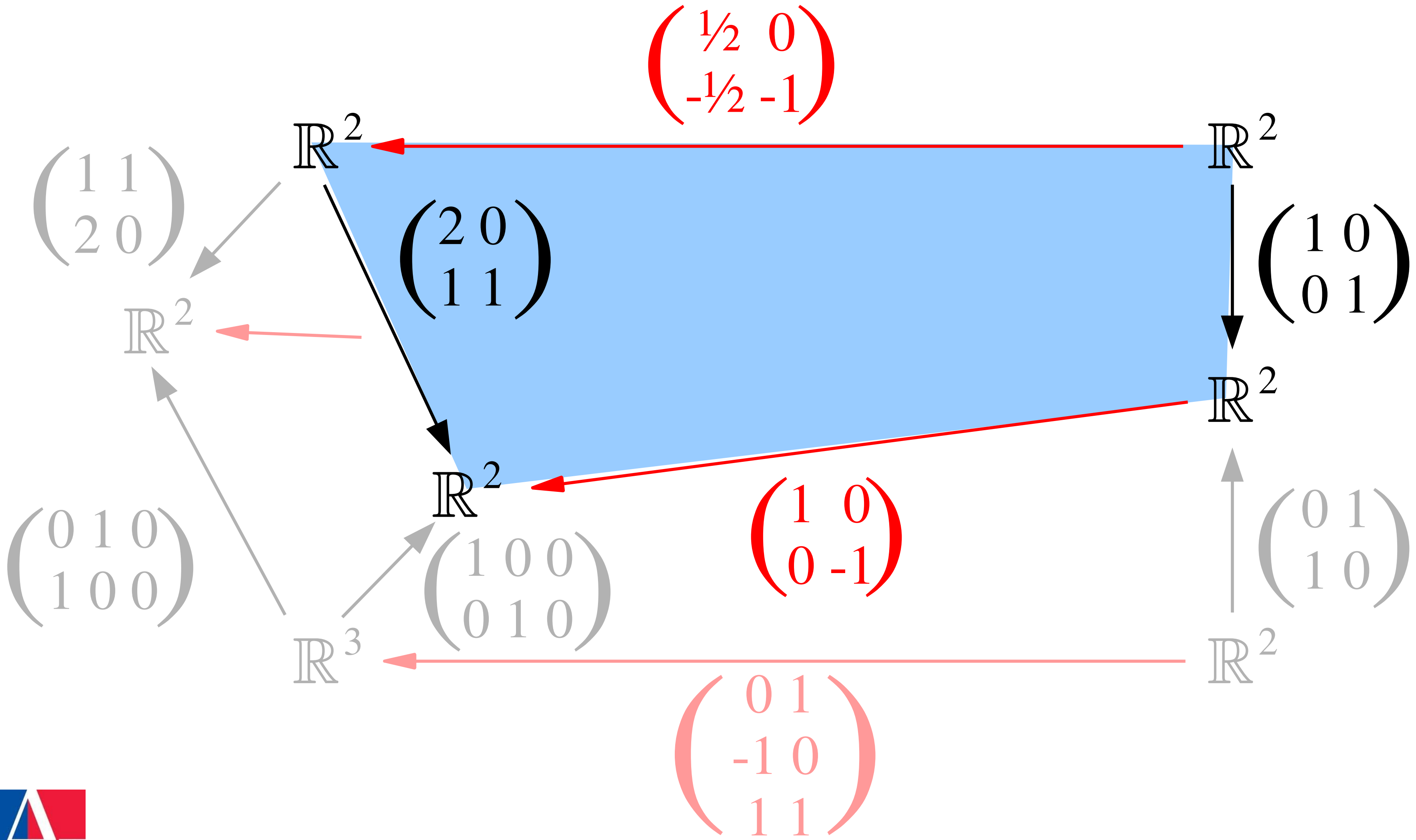
A *sheaf morphism* …

- … and relates them through linear maps …



A *sheaf morphism* ...

- … so the diagram commutes!



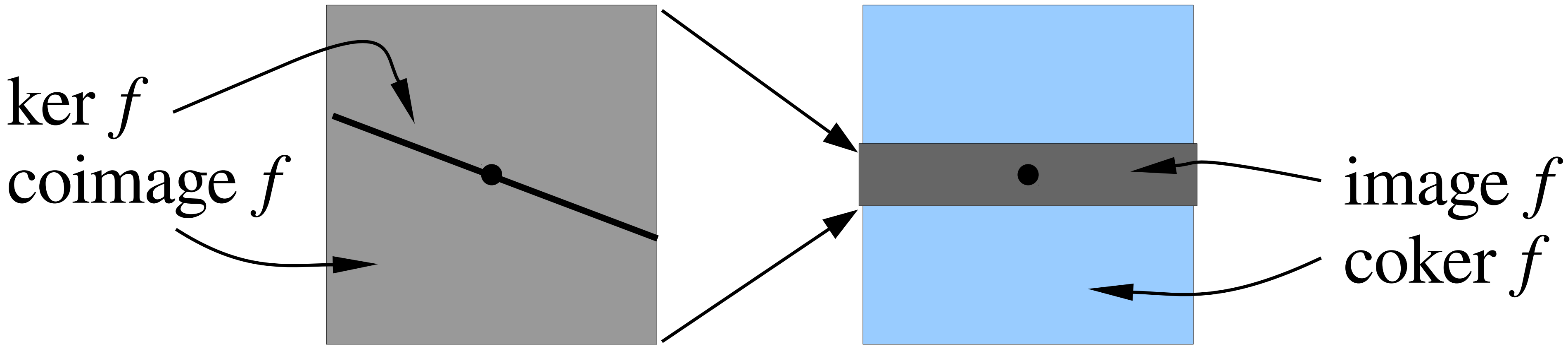
Michael Robinson

An algebraic interlude

The dimension theorem

Theorem: Linear maps between vector spaces are characterized by four fundamental subspaces

$$A \xrightarrow{f} B$$

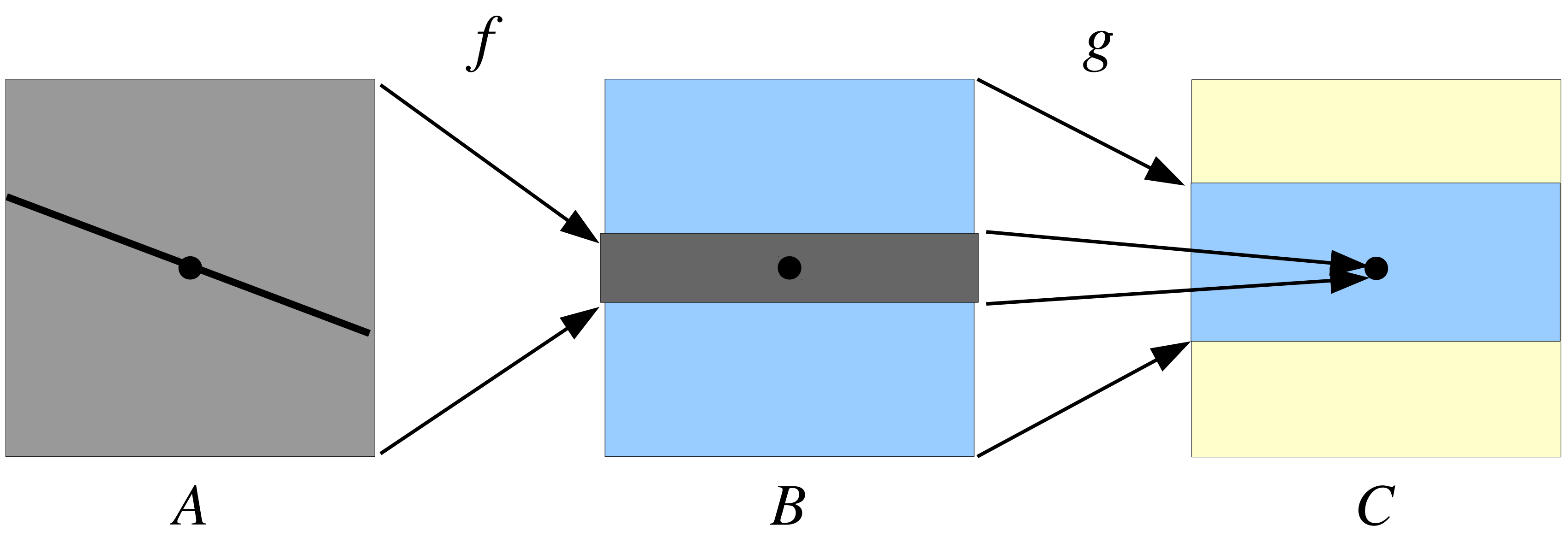


Exactness of a sequence

Exactness of a sequence of maps,

$$A \xrightarrow{f} B \xrightarrow{g} C$$

means that image f = ker g



Example: exact sequence

$$0 \xrightarrow{\begin{pmatrix}0\\0\end{pmatrix}} \mathbb{R}^2 \xrightarrow{\begin{pmatrix}1&0\\0&0\\0&1\\0&0\end{pmatrix}} \mathbb{R}^4 \xrightarrow{\begin{pmatrix}0&1&0&0\\0&0&0&1\\0&1&0&1\end{pmatrix}} \mathbb{R}^3 \xrightarrow{(1\ 1\ -1)} \mathbb{R} \xrightarrow{(0)} 0$$

Properties of exact sequences

Exactness encodes useful properties of maps

- Injectivity

$$0 \to A \xrightarrow{f} B$$

- Surjectivity

$$A \xrightarrow{f} B \to 0$$

- Isomorphism

$$0 \to A \xrightarrow{f} B \to 0$$

- Quotient

$$0 \to A \to B \to B / A \to 0$$

Chain complexes

Exactness is special and delicate. Usually our sequences satisfy a weaker condition:

A *chain complex*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

satisfies image $f \subseteq \ker g$ or equivalently $g \circ f = 0$

Exact sequences are chain complexes, but not conversely

Homology measures the difference

Homology of a chain complex

Starting with a *chain complex*

$$\rightarrow V_{k+1} \xrightarrow{d_{k+1}} V_k \xrightarrow{d_k} V_{k-1} \xrightarrow{d_{k-1}} V_{k-2}$$

Homology is defined as

$$H_k = \ker d_k \,/\, \mathrm{image}\ d_{k+1}$$

All the vectors that are annihilated in stage k ...

… that weren't already present in stage $k + 1$

Homology is trivial if and only if the chain complex is exact

Exact sequences of sheaves

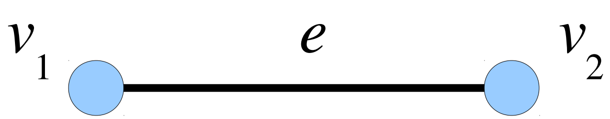
- Similar to vector spaces, we can also have sequences of sheaves with sheaf morphisms between them
- Such a sequence is *exact* if it's exact on **stalks**

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}"] & \mathbb{R}^2 \arrow[r, "\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}"] & \mathbb{R}^2 \\
\mathbb{R}^2 \arrow[u, "\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}"'] \arrow[r, "\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 1 \end{pmatrix}"'] & \mathbb{R}^3 \arrow[u, "\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}"'] & \mathbb{R}^2 \arrow[u, "\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"']
\end{tikzcd}$$

What is **cellular** sheaf cohomology?

Global sections, revisited

- The space of global sections is combinatorially difficult to compute
- It's based on the idea that we can rewrite the basic condition(s) for a global section s of a sheaf $\mathscr{S}$



$$\begin{tikzcd} & \mathscr{S}(e) & \\ \mathscr{S}(v_1) \arrow[ur, "\mathscr{S}(v_1 \rightsquigarrow e)"] & & \mathscr{S}(v_2) \arrow[ul, "\mathscr{S}(v_2 \rightsquigarrow e)"'] \end{tikzcd}$$

$$\mathscr{S}(v_1 \rightsquigarrow e)\, s(v_1) = \mathscr{S}(v_2 \rightsquigarrow e)\, s(v_2)$$

$$+ \mathscr{S}(v_1 \rightsquigarrow e)\, s(v_1) - \mathscr{S}(v_2 \rightsquigarrow e)\, s(v_2) = 0$$

$$- \mathscr{S}(v_1 \rightsquigarrow e)\, s(v_1) + \mathscr{S}(v_2 \rightsquigarrow e)\, s(v_2) = 0$$

($\mathscr{S}(a \rightsquigarrow b)$ is the restriction map connecting cell a to a cell b in a sheaf $\mathscr{S}$)

Global sections, revisited

- The space of global sections is combinatorially difficult to compute
- It's based on the idea that we can rewrite the basic condition(s) for a global section s of a sheaf $\mathscr{S}$

$$\mathscr{S}(v_1 \rightsquigarrow e)\, s(v_1) = \mathscr{S}(v_2 \rightsquigarrow e)\, s(v_2)$$

$$\begin{pmatrix} +\mathscr{S}(v_1 \rightsquigarrow e) & -\mathscr{S}(v_2 \rightsquigarrow e) \end{pmatrix} \begin{bmatrix} s(v_1) \\ s(v_2) \end{bmatrix} = 0$$

$$+\,\mathscr{S}(v_1 \rightsquigarrow e)\, s(v_1) - \mathscr{S}(v_2 \rightsquigarrow e)\, s(v_2) = 0$$

$$-\,\mathscr{S}(v_1 \rightsquigarrow e)\, s(v_1) + \mathscr{S}(v_2 \rightsquigarrow e)\, s(v_2) = 0$$

($\mathscr{S}(a \rightsquigarrow b)$ is the restriction map connecting cell a to a cell b in a sheaf $\mathscr{S}$)

A queue as a sheaf

- Contents of the shift register at each timestep
- $N = 3$ shown

$$\rightarrow \mathbb{R}^2 \xleftarrow[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}]{} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} \mathbb{R}^2 \xleftarrow[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}]{} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} \mathbb{R}^2 \xleftarrow[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}]{} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} \mathbb{R}^2 \xleftarrow[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}]{} \mathbb{R}$$

A single timestep

- Contents of the shift register at each timestep
- $N = 3$ shown

$$\begin{tikzcd}[column sep=small]
{} \arrow[r] & (9,2) & (1,9,2) \arrow[l, "\begin{pmatrix} 0&1&0 \\ 0&0&1 \end{pmatrix}"'] \arrow[r, "\begin{pmatrix} 1&0&0 \\ 0&1&0 \end{pmatrix}"] & (1,9) & (1,1,9) \arrow[l, "\begin{pmatrix} 0&1&0 \\ 0&0&1 \end{pmatrix}"'] \arrow[r, "\begin{pmatrix} 1&0&0 \\ 0&1&0 \end{pmatrix}"] & (1,1) & (5,1,1) \arrow[l, "\begin{pmatrix} 0&1&0 \\ 0&0&1 \end{pmatrix}"'] \arrow[r, "\begin{pmatrix} 1&0&0 \\ 0&1&0 \end{pmatrix}"] & (5,1) & (2,5 \arrow[l, "\begin{pmatrix} 0&1&0 \\ 0&0&1 \end{pmatrix}"']
\end{tikzcd}$$

Rewriting using matrices

- Same section, but the condition for verifying that it's a section is now written linear algebraically

$$
\begin{tikzcd}
(1,9,2) \arrow[r, "\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}"] & (1,9) & (1,1,9) \arrow[l, "\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}"'] \arrow[r, "\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}"] & (1,1) & (5,1,1) \arrow[l, "\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}"']
\end{tikzcd}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix} 1 \\ 9 \\ 2 \\ 1 \\ 1 \\ 9 \\ 5 \\ 1 \\ 1 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

The *cellular cochain complex*

- Motivation: Sections being in the kernel of matrix suggests a higher dimensional construction exists!
- Goal: the *cellular cochain complex* for a sheaf $\mathscr{S}$

$$\cdots ; \mathscr{S}) \xrightarrow{d^{k-1}} \check{C}^k(X; \mathscr{S}) \xrightarrow{d^k} \check{C}^{k+1}(X; \mathscr{S}) \xrightarrow{d^{k+1}} \check{C}^{k+2}(X; \cdots$$

- *Cellular sheaf cohomology* will be defined as

$$\check{H}^k(X; \mathscr{S}) = \ker d^k \,/\, \mathrm{image}\ d^{k-1}$$

much the same as homology (but the chain complex goes up in dimension instead of down)

Generalizing up in dimension

- Global sections lie in the kernel of a particular matrix
- We gather the domain and range from stalks over vertices and edges... These are the *cochain spaces*

$$\check{C}^k(X;\mathscr{S}) = \bigoplus \mathscr{S}(a)$$

a is a *k*-simplex

Note: Ignore any *k*-simplices that are missing faces in *X*

- An element of $\check{C}^k(X;\mathscr{S})$ is called a *cochain*, and specifies a datum from the stalk at each *k*-simplex

(The *direct sum* operator $\bigoplus$ forms a new vector space by concatenating the bases of its operands)

The cellular cochain complex

- The *coboundary map* $d^k : \check{C}^k(X;\mathscr{S}) \to \check{C}^{k+1}(X;\mathscr{S})$ is given by the block matrix

$$d^k = \begin{array}{cc} \begin{pmatrix} & & \\ - - - - & [b_i : a_j]\,\mathscr{S}(a_j \rightsquigarrow b_i) & - - - - \\ & & \end{pmatrix} & \text{Row } i \\ \text{Column } j & \end{array}$$

0, +1, or -1 depending on the relative orientation of a_j and b_i

The cellular cochain complex

- We've obtained the *cellular cochain complex*

$$\cdots;\mathscr{S}) \xrightarrow{d^{k-1}} \check{C}^k(X;\mathscr{S}) \xrightarrow{d^k} \check{C}^{k+1}(X;\mathscr{S}) \xrightarrow{d^{k+1}} \check{C}^{k+2}(X;\cdots$$

- *Cellular sheaf cohomology* is defined as

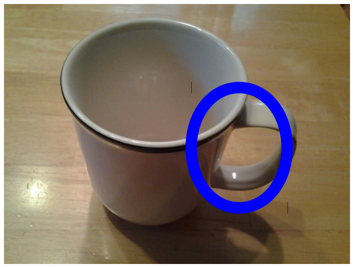
$$\check{H}^k(X;\mathscr{S}) = \ker d^k \,/\, \text{image } d^{k-1}$$

All the cochains that are consistent in dimension k ...

… that weren't already present in dimension k - 1

Cohomology facts

- $\check{H}^0(X;\mathscr{S})$ is the space of global sections of $\mathscr{S}$
- $\check{H}^1(X;\mathscr{S})$ usually has to do with oriented, non-collapsible **data** loops



Nontrivial $\check{H}^1(X;\mathbb{Z})$

- $\check{H}^k(X;\mathscr{S})$ is a functor: sheaf morphisms induce linear maps between cohomology spaces

What is (actual) sheaf cohomology?

(which works even if your space isn't a simplicial complex)

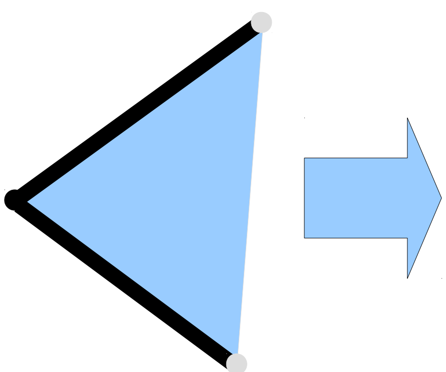
Čech resolution

- The cellular cochain complex can also be viewed as an exact sequence of sheaves: the *Čech resolution*
- Lemma: If U is a star over a point (simplex), then

$$0 \longrightarrow \mathscr{S}(U) \longrightarrow \check{C}^0(U;\mathscr{S}) \longrightarrow \check{C}^1(U;\mathscr{S}) \longrightarrow \check{C}^2(U;\mathscr{S}$$

is an exact sequence (of vector spaces)

Proof: (sketch) Note: stars make this much easier than the usual proof!



$$\begin{tikzcd}
 & \mathscr{S}(\{v_1,v_2\}) \arrow[dr] & \\
\mathscr{S}(\{v_1\}) \arrow[ur] \arrow[dr] \arrow[rr, bend left, "+"] \arrow[rr, bend right, "-"'] & & \mathscr{S}(\{v_1,v_2,v_3\}) \\
 & \mathscr{S}(\{v_1,v_3\}) \arrow[ur] &
\end{tikzcd}$$

Godement resolution

- What if you don't know how to organize the stalks by dimension?
- Well, you basically have to lump everything together

If you have a simplicial complex…

$$0 \rightarrow \mathscr{S}(U) \rightarrow \check{C}^0(U;\mathscr{S}) \rightarrow \check{C}^1(U;\mathscr{S}) \rightarrow \check{C}^2(U;$$

… and if you don't

$$0 \rightarrow \mathscr{S}(U) \rightarrow C^0(U;\mathscr{S}) = \bigoplus \mathscr{S}(a)$$

a is a star over a single point in U

Projecting out stars over simplices of different dimensions

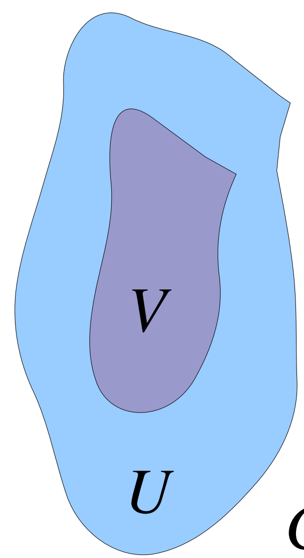
Godement resolution

- Assuming U is a star over a single point, we have only one step of an exact sequence

$$0 \longrightarrow \mathscr{S}(U) \xrightarrow{e} C^0(U;\mathscr{S})$$

All the stalks of $\mathscr{S}$ over points in U

$C^0(\mathscr{S})$ is a sheaf, in which the restriction maps are projections!



$$C^0(V;\mathscr{S}) = \mathscr{S}(V)$$

$$\uparrow \mathrm{pr}_2$$

$$C^0(U;\mathscr{S}) = \mathscr{S}(U) \oplus \mathscr{S}(V)$$

Godement resolution

- We have only one step of an exact sequence, so construct another **via** exactness!

$$0 \longrightarrow \mathscr{S}(U) \xrightarrow{e} C^0(U;\mathscr{S}) \longrightarrow Z^1(U;\mathscr{S}) \longrightarrow 0$$

All the stalks of $\mathscr{S}$ over points in U

Cokernel of e

Basically all the stuff in $C^0(U;\mathscr{S})$ that aren't sections over U

This is a sheaf $Z^1(\mathscr{S})$

Godement resolution

- But $Z^1(\mathscr{S})$ is also a sheaf, so we can reuse our trick!

$$0 \longrightarrow \mathscr{S}(U) \xrightarrow{e} C^0(U;\mathscr{S}) \longrightarrow Z^1(U;\mathscr{S}) \longrightarrow 0$$

$$0 \longrightarrow Z^1(U;\mathscr{S}) \longrightarrow C^0(U;Z^1(\mathscr{S}))$$
$$=: C^1(U;\mathscr{S})$$

All the stalks of $Z^1(U;\mathscr{S})$
over points in U

Godement resolution

- Giving us another step in our sequence!

$$0 \longrightarrow \mathscr{S}(U) \xrightarrow{e} C^0(U; \mathscr{S}) \longrightarrow C^1(U; \mathscr{S})$$

$$C^0(U; Z^1(\mathscr{S}))$$
$$=: C^1(U; \mathscr{S})$$

All the stalks of $Z^1(U; \mathscr{S})$
over points in U

Michael Robinson

Godement resolution

- And repeat!

$$0 \longrightarrow \mathscr{S}(U) \xrightarrow{e} C^0(U;\mathscr{S}) \longrightarrow C^1(U;\mathscr{S}) \longrightarrow Z^2(U;\mathscr{S}$$

$$0 \longrightarrow Z^2(U;\mathscr{S}) \longrightarrow C^0(U;Z^2(\mathscr{S}))$$

$$=: C^2(U;\mathscr{S})$$

All the stalks of $Z^2(U;\mathscr{S})$ over points in U

Godement resolution

- And repeat!

$$0 \longrightarrow \mathscr{S}(U) \xrightarrow{e} C^0(U;\mathscr{S}) \longrightarrow C^1(U;\mathscr{S}) \longrightarrow C^2(U;\mathscr{S}$$

- We obtain an exact sequence of sheaves

General sheaf cohomology

- If U isn't a star over a single point, this probably isn't an exact sequence…

$$0 \rightarrow \mathscr{S}(U) \xrightarrow{e} C^0(U;\mathscr{S}) \rightarrow C^1(U;\mathscr{S}) \rightarrow C^2(U;\mathscr{S}$$

- Theorem: The homology of this resolution is the same as the cellular sheaf cohomology we computed for the simplicial complex!
- But it works even if the base space isn't a simplicial complex

General sheaf cohomology ≅ cellular sheaf cohomology

<u>Proof</u>: We build what's called a *chain morphism* between the two resolutions.

This chain morphism is homotopic to the identity chain morphism, and so preserves homology

$$
\begin{tikzcd}
0 \arrow[r] & \mathscr{S}(U) \arrow[r] & C^0(U;\mathscr{S}) \arrow[r] & C^1(U;\mathscr{S}) \arrow[r] & C^2(U;\mathscr{S}) \\
0 \arrow[r] & \mathscr{S}(U) \arrow[r] \arrow[u, "="] & \check{C}^0(U;\mathscr{S}) \arrow[r] \arrow[u] & \check{C}^1(U;\mathscr{S}) \arrow[r] & \check{C}^2(U;\mathscr{S})
\end{tikzcd}
$$

$$
\begin{array}{r|c|}
 & \text{Stalks of } \mathscr{S} \text{ over vertices} \\
\hline
\text{Stalks of } \mathscr{S} \text{ over vertices} & \begin{matrix} 1 & 0 & \dots & & \dots & 0 \\ 0 & 1 & \dots & \text{identity map} & \dots & 0 \\ 0 & 0 & \dots & & \dots & 1 \end{matrix} \\
\hline
\text{Stalks of } \mathscr{S} \text{ over edges} & \text{Restriction maps to edges} \\
\hline
\dots & \text{More restriction maps} \\
\end{array}
$$

General sheaf cohomology ≅ cellular sheaf cohomology

Proof: We build what's called a *chain morphism* between the two resolutions.

This chain morphism is homotopic to the identity chain morphism, and so preserves homology

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{S}(U) \arrow[r] & C^0(U;\mathscr{S}) \arrow[r] & C^1(U;\mathscr{S}) \arrow[r] & C^2(U; \\
0 \arrow[r] & \mathscr{S}(U) \arrow[r] \arrow[u, "="] & \check{C}^0(U;\mathscr{S}) \arrow[r] \arrow[u] & \check{C}^1(U;\mathscr{S}) \arrow[r] \arrow[ul, dashed] & \check{C}^2(U;
\end{tikzcd}$$

Claim: The C^0 sheaf is *injective*, which allows us to construct another map...

General sheaf cohomology ≅ cellular sheaf cohomology

Proof: We build what's called a *chain morphism* between the two resolutions.

This chain morphism is homotopic to the identity chain morphism, and so preserves homology

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{S}(U) \arrow[r] & C^0(U;\mathscr{S}) \arrow[r] & C^1(U;\mathscr{S}) \arrow[r] & C^2(U;\mathscr{S} \\
0 \arrow[r] & \mathscr{S}(U) \arrow[r] \arrow[u, "="] & \check{C}^0(U;\mathscr{S}) \arrow[r] \arrow[u] & \check{C}^1(U;\mathscr{S}) \arrow[r] \arrow[u] \arrow[ul, dashed, red] & \check{C}^2(U;\mathscr{S}
\end{tikzcd}$$

… which by composing with the top resolution allows us to iterate the process...

General sheaf cohomology ≅ cellular sheaf cohomology

<u>Proof</u>: We build what's called a *chain morphism* between the two resolutions.

This chain morphism is homotopic to the identity chain morphism, and so preserves homology

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{S}(U) \arrow[r] & C^0(U;\mathscr{S}) \arrow[r] & C^1(U;\mathscr{S}) \arrow[r] & C^2(U;\mathscr{S}) \\
0 \arrow[r] & \mathscr{S}(U) \arrow[r] \arrow[u, "="] & \check{C}^0(U;\mathscr{S}) \arrow[r] \arrow[u] & \check{C}^1(U;\mathscr{S}) \arrow[r] \arrow[u] & \check{C}^2(U;\mathscr{S}) \arrow[u] \arrow[ul, dashed]
\end{tikzcd}$$

… which by composing with the top resolution allows us to iterate the process...

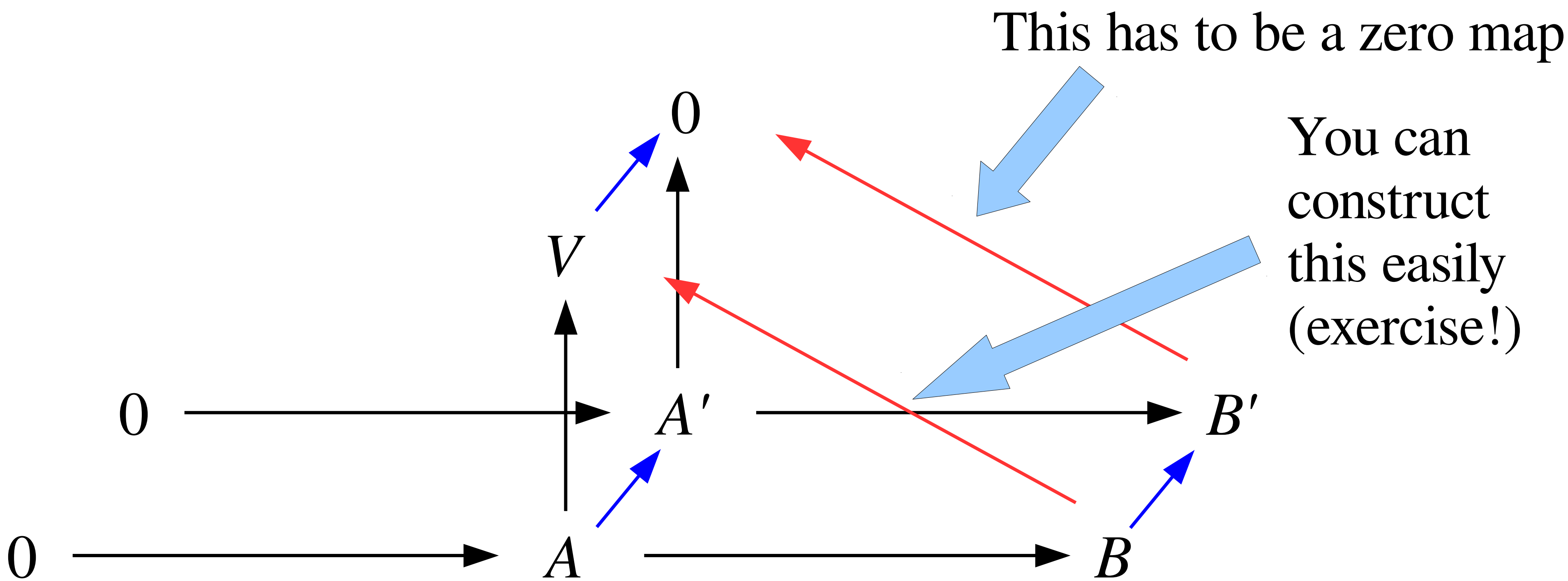
$C^0(\mathscr{S})$ is injective

- What this means is that starting with a diagram of **black** sheaf morphisms like the one below, you can construct the **red** sheaf morphism

$$\begin{tikzcd}
 & C^0(\mathscr{S}) & \\
0 \arrow[r] & A \arrow[u] \arrow[r] & B \arrow[ul]
\end{tikzcd}$$

$C^0(\mathscr{S})$ is injective

- This basically amounts to being able to extend a diagram like so, which is the prototypical sheaf with a projection for a restriction map



<u>Task</u>: check commutativity of the 2 triangles and one square with **red** edges
You're given commutativity of the two squares with **black** and **blue** edges

Coda: an example!

- Consider the sheaf $\mathscr{S}$ over a single edge:

$$\begin{tikzcd} \mathbb{R} \arrow[d, "\begin{bmatrix}1\\0\end{bmatrix}"'] \\ \mathbb{R}^2 \\ \mathbb{R} \arrow[u, "\begin{bmatrix}1\\0\end{bmatrix}"] \end{tikzcd}$$

- Its cellular cochain complex is

$$\check{C}^0(X;\mathscr{S}) \xrightarrow{d^0} \check{C}^1(X;\mathscr{S}) \xrightarrow{d^1} \check{C}^2(X;\mathscr{S}) \xrightarrow{d^2} \check{C}$$

$$\mathbb{R}\oplus\mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow 0 \longrightarrow 0 \longrightarrow$$

- So that $\quad d^0 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$$\check{H}^0(\mathscr{S}) = \ker d^0 \cong \mathbb{R} \qquad \check{H}^1(\mathscr{S}) = \mathbb{R}^2 \,/\, \text{image } d^0 \cong \mathbb{R}$$

Coda: an example!

- Here's the Godement resolution … complicated, but doable!

$$
\begin{tikzcd}
\mathbb{R} \arrow[r] \arrow[d, "\begin{bmatrix}1\\0\end{bmatrix}"'] & \mathbb{R}\oplus\mathbb{R}^2 \arrow[r] \arrow[d, "\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}"'] & \mathbb{R}^2 \arrow[r] \arrow[d] & \mathbb{R}^2 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
\mathbb{R}^2 \arrow[r, "\begin{bmatrix}1&0\\0&1\end{bmatrix}"] & \mathbb{R}^2 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & {} \\
\mathbb{R} \arrow[r, "\begin{bmatrix}1\\1\\0\end{bmatrix}"'] \arrow[u, "\begin{bmatrix}1\\0\end{bmatrix}"] & \mathbb{R}\oplus\mathbb{R}^2 \arrow[r, "\begin{bmatrix}1&-1&0\\0&0&1\end{bmatrix}"'] \arrow[u, "\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}"] & \mathbb{R}^2 \arrow[r, "\begin{bmatrix}1&0\\0&1\end{bmatrix}"'] \arrow[u] & \mathbb{R}^2 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & {}
\end{tikzcd}
$$

$$
\mathscr{S} \rightarrow C^0(\mathscr{S}) \rightarrow Z^1(\mathscr{S}) \rightarrow C^1(\mathscr{S}) \rightarrow C^2(\mathscr{S}) \rightarrow
$$

Coda: an example!

- Here's the Godement resolution … complicated, but doable!

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\begin{bmatrix}1\\1\\0\end{bmatrix}"] \arrow[d, "\begin{bmatrix}1\\0\end{bmatrix}"'] & \mathbb{R}\oplus\mathbb{R}^2 \arrow[r, "\begin{bmatrix}1 & -1 & 0\\0 & 0 & 1\end{bmatrix}"] \arrow[d, "\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}"'] & \mathbb{R}^2 \arrow[r, "\begin{bmatrix}1&0\\0&1\end{bmatrix}"] \arrow[d] & \mathbb{R}^2 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
\mathbb{R}^2 \arrow[r, "\begin{bmatrix}1&0\\0&1\end{bmatrix}"] & \mathbb{R}^2 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & {} \\
\mathbb{R} \arrow[r, "\begin{bmatrix}1\\1\\0\end{bmatrix}"'] \arrow[u, "\begin{bmatrix}1\\0\end{bmatrix}"] & \mathbb{R}\oplus\mathbb{R}^2 \arrow[r, "\begin{bmatrix}1 & -1 & 0\\0 & 0 & 1\end{bmatrix}"'] \arrow[u, "\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}"] & \mathbb{R}^2 \arrow[r, "\begin{bmatrix}1&0\\0&1\end{bmatrix}"'] \arrow[u] & \mathbb{R}^2 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & {}
\end{tikzcd}$$

$$\mathscr{S} \rightarrow C^0(\mathscr{S}) \rightarrow Z^1(\mathscr{S}) \rightarrow C^1(\mathscr{S}) \rightarrow C^2(\mathscr{S}) \rightarrow$$

$$\mathscr{S}(X) \rightarrow C^0(X;\mathscr{S}) \xrightarrow{m} C^1(X;\mathscr{S}) \rightarrow C^2(X;\mathscr{S}$$

$$\mathscr{S}(X) \rightarrow \mathbb{R}\oplus\mathbb{R}^2\oplus\mathbb{R} \xrightarrow{m} \mathbb{R}^2\oplus\mathbb{R}^2 \rightarrow 0 \rightarrow$$

Coda: an example!

- Here's the Godement resolution … complicated, but doable!

$$
\begin{tikzcd}
\mathbb{R} \arrow[r] \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"'] & \mathbb{R}\oplus\mathbb{R}^2 \arrow[r] \arrow[d] & \mathbb{R}^2 \arrow[r] \arrow[d] & \mathbb{R}^2 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
\mathbb{R}^2 \arrow[r, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"'] & \mathbb{R}^2 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & {} \\
\mathbb{R} \arrow[r, "\begin{pmatrix}1\\1\\0\end{pmatrix}"'] \arrow[u, "\begin{pmatrix}1\\0\end{pmatrix}"] & \mathbb{R}\oplus\mathbb{R}^2 \arrow[r, "\begin{pmatrix}1&-1&0\\0&0&1\end{pmatrix}"'] \arrow[u] & \mathbb{R}^2 \arrow[r, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"'] \arrow[u] & \mathbb{R}^2 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & {}
\end{tikzcd}
$$

$$\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix} \qquad \begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}$$

$$\mathscr{S}(X) \longrightarrow \mathbb{R}\oplus\mathbb{R}^2\oplus\mathbb{R} \xrightarrow{m} \mathbb{R}^2\oplus\mathbb{R}^2 \longrightarrow 0 \longrightarrow$$

$$m = \left(\begin{array}{c|cc|c} 1 & -1 & 0 & \\ 0 & 0 & 1 & \\ \hline & -1 & 0 & 1 \\ & 0 & 1 & 0 \end{array}\right) \Rightarrow \text{rank } m = 3 \Rightarrow$$

$$H^0(\mathscr{S}) = \ker m \cong \mathbb{R}$$

$$H^1(\mathscr{S}) = \mathbb{R}^4/\text{image } m \cong \mathbb{R}$$

Conclusion!

- It's possible to compute the cohomology of sheaves over general finite topological spaces
- You can use the Godement resolution, which might be involved, but it's actually a finite calculation
- But if your space is a simplicial complex, then the cellular sheaf cohomology is much easier to compute

For more information

Take MATH 496/696-001

Computational Algebraic Topology

Next semester!

(lots more diagrams will ensue!!!)

Michael Robinson

michaelr@american.edu