What is topology?
Session objectives

• What is a topological space?
  – What kinds of spaces are there?

• How to represent topological spaces effectively for data fusion problems
  – Simplicial complex models, and how they interface with topology

• Encoding data as a topological space
What is topology?

Topology is the study of spaces under continuous deformations.

Topological equivalence:

![Mug](image1.png) = ![Donut](image2.png)
What is topology?

Not this!
This is TopoGRAPHY!

(Thanks USGS!)
Topological imaging

- Cellphone used to record signal level of 802.11 access points near several apartment buildings
- Signal level (dBm) and station MAC address recorded periodically
  - Uniquely identified 52 WAPs
- Random projection to 2d

(image courtesy of Google)
Software credit: Daniel Muellner and Mikael Vejdemo-Johansson
Topological imaging

- Goal: measure environment and targets with minimal sensing and opportunistic sources
- Key theoretical guarantees proven
- First generation algorithms
  - Simulated extensively
  - Validated experimentally

(image courtesy of Google)

Software credit: Daniel Muellner and Mikael Vejdmo-Johansson
What kind of space?

<table>
<thead>
<tr>
<th>General topological space</th>
<th>Cellular space</th>
</tr>
</thead>
<tbody>
<tr>
<td>• More general (obviously!)</td>
<td>• Less general (and less pathological)</td>
</tr>
<tr>
<td>• Very traditional</td>
<td>• Good for computation</td>
</tr>
<tr>
<td>• But admits pathological cases</td>
<td>• Less general (and less expressive)</td>
</tr>
<tr>
<td>Manifolds</td>
<td>• More constraints</td>
</tr>
<tr>
<td>• Very restricted, but this aids estimation</td>
<td></td>
</tr>
<tr>
<td>• Calculus!</td>
<td></td>
</tr>
</tbody>
</table>
Some (mostly) topological spaces

- “is a”
- “can be made into a”

Manifold → Regular CW complex → Cell complex → CW complex → Stratified manifold → Topological space

Simplicial complex → Δ-complex → Flag complex → Abstract simplicial complex

CW complex → Undirected hypergraph

Preorder

Note to category theorists:
This diagram does NOT commute!
Spaces of interest for this tutorial

“is a”
“can be made into a”

Flag complex
Undirected graph

Abstract simplicial complex
Undirected hypergraph
Preorder

Manifold
Cell complex
CW complex
Δ-complex
Regular CW complex
Simplicial complex
Topological space
Stratified manifold

Note to category theorists:
This diagram still does NOT commute!
High level intuition first...

... then the precise details
Simplicial complexes

- A simplicial complex is a collection of vertices and ...

A vertex represents an individual measurement taken by a sensor
Simplicial complexes

- \( \ldots \text{edges} \) (pairs of vertices) and \( \ldots \)

An edge represents the fact that two measurements should be tested for consistency.

Usually people call this a graph; I will too.

NB: I got lazy… \( v_1 \) should be \( \{v_1\} \)
Simplicial complexes

- ... higher dimensional simplices (tuples of vertices)
- Whenever you have a simplex, you have all subsets, called faces, too.

A simplex represents that several measurements should be tested for consistency.
Undirected hypergraphs

- ... might have simplices that are missing some faces
- ... and we call simplices hyperedges as a reminder

A hyperedge still represents that several measurements should be tested for consistency
Flag complexes

- All *cliques* – an edge between every pair of vertices – are simplices, too

\[
\begin{align*}
\{v_1, v_2\} & \quad \{v_2, v_3\} \\
\{v_1, v_3\} & \quad \{v_3, v_4\} \\
\{v_1, v_3\} & \quad \{v_2, v_3\} \\
\{v_2, v_3\} & \quad \{v_3, v_4\} \\
\end{align*}
\]
Helpful taxonomy

- All cliques are simplices
- Simplices are defined by their vertices, simplices always have all their faces
- Hyperedges defined by vertices, but other than that, anything goes...
- Easier to compute invariants
- Harder to compute invariants

- Flag complex
- Abstract simplicial complex
- Undirected hypergraph
- Vertices and edges only

Abstract simplicial complex

Undirected graph

Undirected hypergraph
Now the details!
Abstract Simplicial Complex

Given a set $V$, which we'll call vertices, an abstract simplicial complex is a collection $X$ of subsets of $V$ satisfying the property:

If $a \in X$ and $b \subseteq a$ then $b \in X$.

(If you ignore this property, then $X$ is a hypergraph.)

Elements of $X$ are called simplices

- A simplex $a = \{a_0, a_1, \ldots a_n\}$ is said to have dimension $n$
- A maximal simplex is called a facet or a toplex
- Dimension $0 = vertex$, dimension $1 = edge$
Flag complexes and graphs

- If $X$ only has vertices and edges, it's called a graph
- A clique in a graph is a subgraph in which every pair of vertices has an edge between them
- If all cliques are also simplices, then $X$ is a flag complex

- Flag complexes are often better choices than graphs because they capture higher dimensional (multi-way) relationships
Topology

• A way to describe neighborhoods of simplices
• This is really good for describing where a piece of data is valid – near the sensor, usually
• There's a natural way to do this completely generally, and it's very handy mathematically
  – It leads to computational trickiness, though
  – We'll use the Alexandrov topology when needed (usually for hypergraphs) and avoid the general definition
• Fortunately, there's usually a way to go between topology and simplicial complexes
Topology, precisely

If $X$ is a set, then a topology $T$ on $X$ is a collection of subsets of $X$ satisfying four axioms:

1. The empty set is in $T$
2. $X$ is in $T$
3. If $U$ is in $T$ and $V$ is in $T$ then $U \cap V$ is in $T$
4. All unions of elements of $T$ are in $T$

The elements of $T$ are called the open sets for the topology

The pair $(X, T)$ is called a topological space
Stars over simplices

- The *star* over a simplex is that simplex along with all higher dimensional ones containing it.

Star over \( v_1 \):

\[
\{ v_1 \} \subseteq \{ v_1, v_2, v_3 \} \subseteq \{ v_2, v_3 \} \subseteq \{ v_2, v_3, v_4 \} \subseteq \{ v_3, v_4 \} \subseteq \{ v_4 \}
\]
Stars over simplices

- The *star* over a simplex is that simplex along with all higher dimensional ones containing it.

Star over \( \{v_1, v_2\} \):

\[
\{v_1, v_2\} \cup \{v_1, v_2, v_3\} \cup \{v_2, v_3\} \cup \{v_3, v_4\}
\]
Stars over simplices

- The *star* over a simplex is that simplex along with all higher dimensional ones containing it.

- **Proposition**: The collection of all possible stars and all possible unions of stars forms a topology for the set of simplices.

- It's called the *Alexandrov topology*, and it's usually the correct one for data integration problems.

- This works for hypergraphs, not just abstract simplicial complexes.
Beyond hypergraphs

- Sometimes you need to duplicate simplices…
  - This isn't a problem for the topology, but it violates our combinatorial construction thus far
  - This usually leads to multi-graphs and multi-hypergraphs
- The Alexandrov topology still works, though, and this causes no computational problems in the end
Beyond hypergraphs

- What's important is the set of *attachments* between simplices
  - These form a partial order for the simplices, via set inclusion
  - Vertices are subsets of edges, etc.
Starting with a topology

- There's usually a physical region associated to a sensor where it's gathering information
  - It's usually good practice to assert that this is an open set in some appropriate topology, often the usual topology for Euclidean space
Recall: Multi-INT tracking

- Each sensor has a region associated to it
  - Should be an open set in Euclidean space for these sensors; a good guess for physical sensors
Topologizing the sensor space

- We now know this is a simplicial complex!
- But how did we get it?
Nerve of an open cover

Let \((X,T)\) be a topological space and \(\mathcal{U} = \{U_1, U_2, \ldots\}\) be a collection of open sets whose union is \(X\) (an open cover).

The nerve \(N(\mathcal{U})\) is the abstract simplicial complex with

- The set of vertices given by \(\mathcal{U}\), and
- Whenever \(U_a \cap U_b \cap \ldots\) is nonempty, \(\{U_a, U_b, \ldots\}\) is a simplex in \(N(\mathcal{U})\).

A crucial theorem (Leray theorem) links \(N(\mathcal{U})\) to the topological structure of \((X,T)\).
Next up...

- Lunch!
- Next lecture: Define sheaves!
Further reading...

- Dai Tamaki, “Cellular stratified spaces”