Sheaves and Numerical Analysis

Michael Robinson
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Complex behaviors of dynamical systems

Sonar echos

Econometrics

Improvisational Music

Michael Robinson

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Key points

**Question:** Does discretization destroy dynamical structure?

- Differential equations can be **encoded** as *sheaves*
- *Consistency* of numerical methods is characterized by the **commutativity** of *sheaf morphisms*
- Time evolution induces a **universal** sheaf morphism
The big picture

- Partial orders describe the relationships between variables in a system… order relations correspond to (differential) operators.

- Every partial order has a natural topology, the Alexandroff topology.
  - Presheaves and sheaves “are the same thing” in this topology, since the gluing axiom is satisfied trivially.
  - Commutativity is the only actual constraint on a sheaf diagrams.

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Michael Robinson
Topologizing a partial order
Topologizing a partial order

Open sets are unions of \textit{up-sets}
Topologizing a partial order

Open sets are unions of *up-sets*
Topologizing a partial order

*Closed sets* are complements of open sets
Topologizing a partial order

Intersections of up-sets are also up-sets
Topologizing a partial order

Intersections of up-sets are also up-sets
A sheaf on a poset is...

A set assigned to each element, called a stalk, and …

(The stalk on an element in the poset is better thought of being associated to the up-set)

This is a sheaf of vector spaces on a partial order.
A sheaf on a poset is...

… restriction functions between stalks, following the order relation...

("Restriction" because it goes from bigger up-sets to smaller ones)

This is a sheaf of vector spaces on a partial order
A sheaf on a poset is...

… so that the diagram commutes!

This is a sheaf of vector spaces on a partial order
An assignment is...

... the selection of a value from all stalks

The term *serration* is more common, but perhaps more opaque.
A global section is...

... an assignment that is consistent with the restrictions

\[
\begin{pmatrix}
  0 & 1 & 1 \\
  1 & 0 & 1 \\
  1 & 0 & 1
\end{pmatrix}
\]
Some assignments aren’t consistent

... but they might be partially consistent
Goodwin macroeconomic model

- A simple description of a national economy:

\[ \dot{v} = v(t) \left( \frac{1}{\sigma} - (\alpha + \beta) - \frac{u(t)}{\sigma} \right) \quad (1) \]
\[ v = \text{Employment rate} \]

\[ \dot{u} = u(t) \left( - (\alpha + \gamma) + (\rho v(t)) \right). \quad (2) \]
\[ u = \text{Workers’ share of income} \]
Goodwin macroeconomic model

- A simple description of a national economy:

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\[ \dot{v} = dv/dt \]  \hspace{1cm} (3)  \\
\[ \dot{u} = du/dt \]  \hspace{1cm} (4)
Goodwin macroeconomic model

- A simple description of a national economy:

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(1) \[ v = \text{Employment rate} \]

\[ \dot{u} = u(t) (- (\alpha + \gamma) + (\rho v(t))) \]. \]  
(2) \[ u = \text{Workers’ share of income} \]

\[ \dot{v} = d\frac{v}{dt} \]  
(3)

\[ \dot{u} = d\frac{u}{dt} \]  
(4)
Multi-equation sheaves

- **Theorem**: (R.) For every system of equations, there is a sheaf whose global sections are solutions
  - Base poset has two levels: Equations < Variables
  - Stalk over each variable is that variable’s set of possible values
  - Stalk over an equation is a subset of the product of the variables involved
  - Restriction maps are projections

**Proof**: Straightforward once the construction is built!

Consistency: Discretizing correctly
Discretization of functions

\[ C^k(X,Y) \rightarrow \mathbb{R}^n \]

\[ f \rightarrow (f(x_1), \ldots, f(x_n)) \]
Discretization of functions

\[ \mathbb{R}^m \rightarrow C^k(X,Y) \rightarrow \mathbb{R}^n \]

\[ (a_1, \ldots, a_m) \rightarrow f = \sum a_i f_i(x) \]
Why discretize?

\[ \mathbb{R}^m \rightarrow C^k(X,Y) \rightarrow \mathbb{R}^n \]

\[ C^k(X,Y) \]

differential operator

\[ C^k-p(X,Y) \]
Why discretize?

\[ \mathbb{R}^m \rightarrow C^k(X, Y) \rightarrow \mathbb{R}^n \]

\[ \mathbb{R}^m \rightarrow C^k-p(X, Y) \rightarrow \mathbb{R}^n \]

differential operator
Why discretize?

**Goals:**
1. Make the diagram commute as $m, n \to \infty$ (consistency of the approximation)
2. Recover properties of the differential operator from the approximations (convergence of the approximation)
Goal: a sheaf interpretation

\[ \mathbb{R}^m \rightarrow C^k(X,Y) \rightarrow \mathbb{R}^n \]

finite element approx

differential operator

\[ \mathbb{R}^m \rightarrow C^k - p(X,Y) \rightarrow \mathbb{R}^n \]

differential operator

finite difference approx

approx sheaf morphism

approx sheaf morphism

C \rightarrow S \rightarrow D

sheaf

sheaf

sheaf

Encoding
A simple example

- Consider \( u' = f(u) \) on the real line
- This has a sheaf diagram

\[
\begin{array}{c}
\mathcal{C}^0(\mathbb{R}, \mathbb{R}^d) \\
\mathcal{C}^1(\mathbb{R}, \mathbb{R}^d)
\end{array}
\begin{array}{c}
\mathcal{C}^1(\mathbb{R}, \mathbb{R}^d) \\
\mathcal{C}^1(\mathbb{R}, \mathbb{R}^d)
\end{array}
\begin{array}{c}
\xrightarrow{f} \\
\xleftarrow{\text{id}} \\
\xrightarrow{\frac{d}{dt}} \\
\xleftarrow{\text{id}}
\end{array}
\]
Finite differences

- Discretizing each function space via a fixed step $h$

$$(\Delta_h u)_n = u(nh)$$

Continuous sheaf model

$C^0(\mathbb{R}, \mathbb{R}^d) \xrightarrow{id} C^1(\mathbb{R}, \mathbb{R}^d) \xrightarrow{id} (\mathbb{R}^d)_{\mathbb{Z}}$

$C^1(\mathbb{R}, \mathbb{R}^d) \xrightarrow{id} C^1(\mathbb{R}, \mathbb{R}^d) \xrightarrow{id} (\mathbb{R}^d)_{\mathbb{Z}}$

Discretized sheaf model

$\Delta_h$ $\Delta_h$

$d/dt$ $D_h$
Is it a sheaf morphism?

- A sheaf morphism is a commutative diagram specified by the dotted arrows… is this one?

Continuous sheaf model

Discretized sheaf model
Is it a sheaf morphism?

• This square commutes if we pick \( \tilde{f} \) correctly...
Is it a sheaf morphism?

• … this one commutes trivially …

\[ C^0(\mathbb{R}, \mathbb{R}^d) \xrightarrow{f} C^1(\mathbb{R}, \mathbb{R}^d) \xleftarrow{id} C^1(\mathbb{R}, \mathbb{R}^d) \xrightarrow{d/dt} C^1(\mathbb{R}, \mathbb{R}^d) \]

\[ \Delta_h \]

\[ (\mathbb{R}^d)^\mathbb{Z} \xrightarrow{\sim f} (\mathbb{R}^d)^\mathbb{Z} \xleftarrow{id} (\mathbb{R}^d)^\mathbb{Z} \xrightarrow{D_h} (\mathbb{R}^d)^\mathbb{Z} \]

Continuous sheaf model  Discretized sheaf model
Is it a sheaf morphism?

• … this one also commutes trivially …
Is it a sheaf morphism?

- ...but this asks that $u'(nh) = D_hu_n$, which means discretized version is exactly correct. Oops!

Continuous sheaf model

Discretized sheaf model
Finite elements

- We can also try to construct a finite elements approximation... from the “other side”
- Again start with the same continuous sheaf model

\[
\begin{array}{ccc}
C^0(\mathbb{R},\mathbb{R}^d) & \xleftarrow{f} & C^1(\mathbb{R},\mathbb{R}^d) \\
\xrightarrow{id} & & \xrightarrow{id} \\
C^1(\mathbb{R},\mathbb{R}^d) & \xrightarrow{d/dt} & C^1(\mathbb{R},\mathbb{R}^d)
\end{array}
\]
Finite elements sheaf model

- Use an $N$ dimensional subspace of functions with a linear embedding $b : \mathbb{R}^N \rightarrow B \subseteq C^1(\mathbb{R}, \mathbb{R}^d)$. 
Is it a sheaf morphism?

- Although the derivative approximation can now be corrected by a judicious choice of embedding $b$...
Might be a sheaf morphism...

- ...if not linear, now the equation itself fails
- ...if linear, we may get a morphism; Galerkin method!
Observations about consistency

$\mathbb{R}^m \rightarrow C^k(X,Y) \rightarrow \mathbb{R}^n$

If linear, can be exact!

$C^k - p(X,Y) \rightarrow \mathbb{R}^n$

encoding

approx sheaf morphism

defect in approximating the equation

approx defect in approximating derivatives
Convergence: Behavior of solutions
Global sections and dynamics

- Notice that $u' = f(u)$ is autonomous
- Thus global sections of are invariant under the action of time translation... can we generalize?
Dynamics on sheaves

- Sheaf $S$ and a diffeomorphism $f : S(X) \rightarrow S(X)$ on its space of global sections.

- Does it extend to a sheaf automorphism?
  - In our simple example, it does!
  - In general, though, it may not!

- **Conjecture**: there is a cohomological obstruction

- But I do have a lead...
Sheaf dynamics theorem

- Sheaf $S$ and a diffeomorphism $f : S(X) \to S(X)$ on its space of global sections.

- **Theorem: (R.)** There is a (possibly different) sheaf $R$ with the same (or more) global sections as $S$
  - There is a sheaf morphism $F : S \to R$ that induces $f$ on global sections
  - $R$ is universal: any other such sheaf $P$ factors through $R$
Proof technique: pushouts

- First, construct the stalks and component maps...

\[ S(X) \rightarrow S(U) \rightarrow R(U) \]
Proof technique: pushouts

• … then construct the restrictions

\[
\begin{array}{c}
S(X) \\
S(U) \\
S(V) \\
R(U) \\
R(V)
\end{array}
\]
Proof technique: pushouts

• … then construct the restrictions

• More technical details: gluing, universality...
Next steps: analysis!

- When does a dynamical system induce a sheaf automorphism?
- Now we understand part of the diagram… but how does it all fit together?
- Can we push out along approximate morphisms?
For more information

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