Topological Symmetries: Quasiperiodicity and its Application to Filtering and Classification Problems

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- Students:
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  - Jen Dumiak
  - Sean Fennell

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- Website reference
  http://www.drmichaelrobinson.net/
The main idea

- Factoring out a smooth map from a signal may reveal a group action; **denoise on the quotient by this action**

- **Theorem**: (R.) There is an optimal, data-driven choice of domain that characterizes all symmetries in the signature

Sonar input data format

Range →

Echo strength

Pulse number →
Goal: reorganize and denoise!

- Echo strength
- Varying degrees of errors
Goal: reorganize and denoise!

Partially known (but probably redundant) topology

Known geometry

Range →

Pulse number

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Circular coordinates

- Given $u(x)$, obtain
  \[ P(x) = (u(x), u(x + x_1), u(x + x_2), \ldots, u(x + x_n)) \]

- **Theorem**: (Takens) Almost every $P$ is an embedding for sufficiently large $n$ and generic choice of $x_k$

- So if a function is periodic, the image of $P$ is a circle

- If $u$ is not periodic but the image of $P$ remains close to a circle (not a helix), we're still in good shape
  - Persistent cohomology* can compute smooth phase functions from time delay maps

Topology > Pulse order in simulation

- Since the topology of the sensor space constrains the embedding, pulse order can be recovered if unknown.

1000 random azimuthal looks at 5 point scatterers

(First 3 principal components of the data)
Topology > Pulse order in practice, too

Raw time samples

Pulses organized into a phase space

Re-examining periodic functions

- **Via symmetry**: A function \( u : \mathbb{R} \to \mathbb{R} \) is *periodic* if there exists a \( T \) such that
  \[
  u(x) = u(x + T) \quad \text{for all } x
  \]
- **Via diagrams**: Periodic functions factor through the circle:

  The *phase function* \( \phi(x) = 2\pi \left(\frac{x}{T}\right) \mod 1 \)
Re-examining periodic functions

- **Via symmetry:** A function \( u : \mathbb{R} \to \mathbb{R} \) is *periodic* if there exists a \( T \) such that \( u(x) = u(x + T) \) for all \( x \)

- **Via diagrams:** Periodic functions factor through the circle:

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{u} & \mathbb{R} \\
\phi \downarrow & & \downarrow U \\
S^1 & & \\
\end{array}
\]

The *phase function* can be generalized...

...as can the *phase space*
Generalizing beyond circular

- Certainly, we should ask for more general inputs and outputs... manifolds are good (want calculus)
- Then we should assume $u, \phi, U$ are all smooth

We'd like an analog of a monotonic function here: quotient map
Why manifolds?

- If the phase space is not required to be a manifold, then the best choice is the topological quotient $M / u$

This has an annoying consequence... $\phi$ can have critical points
Accidental contractibility

- Problem: The phase space isn't amenable to cohomological periodicity detection using $H^1$
- Consider $u(x) = \sin x$, then the factorization looks like

\[
\begin{align*}
\mathbb{R} & \xrightarrow{\sin} \mathbb{R} \\
[-\pi/2, \pi/2] & \xrightarrow{\phi} \mathbb{R}
\end{align*}
\]

Contractible phase space

… so we'd better ensure $\phi$ has constant rank
Quasiperiodicity

- **Definition:** a smooth function \( u \) has a *quasiperiodic factorization* given by the commutative diagram below when \( \phi \) is a surjective submersion

A consequence of \( \phi \) being a surjective submersion is that \( C \) is a manifold

\[
\begin{array}{ccc}
M & \xrightarrow{u} & N \\
\downarrow{\phi} & & \downarrow{U} \\
C & \xrightarrow{U} & C \\
\end{array}
\]

- We'll say \( u \) is \((\phi,C)\)-quasiperiodic in this case
Theorem: Optimal compressed signatures always exist
A quasiperiodic factorization

• Consider \( u \): given by
  \[
  u(x) = \sin x
  \]

• Here's a quasiperiodic factorization
  \[
  \phi(x) = x \mod 2\pi
  \]
  \[
  U(x) = \sin x
  \]

\[
M = \mathbb{R} \xrightarrow{u} N = \mathbb{R}
\]

\[
\phi
\]

\[
\phi
\]

\[
U
\]

\[
C = S^1
\]

\[
0 = 2\pi
\]
A quasiperiodic factorization

- Consider $u$: given by
  
  $u(x) = \sin x$

- Here's another quasiperiodic factorization
  
  $\phi'(x) = (x / 2) \mod 2\pi$

  $U'(x) = \sin 2x$

\[ \begin{align*}
M &= \mathbb{R} \\
N &= \mathbb{R} \\
C &= S^1
\end{align*} \]
Factorizations can be weird

- Consider $u: \mathbb{R} \to S^1$ given by $u = U \circ \phi$ where

  $\phi : \mathbb{R} \to S^1$, given by $\phi(x) = (6 \arctan x) \mod 2\pi$

  $U : S^1 \to S^1$, given by $U(x) = x$

  - This is a quasiperiodic factorization, but the function doesn't "repeat" -- every point in the range has finitely many preimages
Non-uniqueness of factorizations

• The category \textbf{QuasiP}(u) for a smooth function \(u\):
  
  – Objects: quasiperiodic factorizations \((\phi, U)\)
  
  – Morphisms: \((\phi, U) \rightarrow (\phi', U')\) if there's a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\phi} & C \\
\downarrow{\phi'} & & \downarrow{U} \\
C' & \xrightarrow{U'} & N
\end{array}
\]

• **Theorem:** (R.) \textbf{QuasiP}(u) has a unique final object, called the \textit{universal quasiperiodic factorization} of \(u\)

  – It's the correct phase space for a topological filter tuned to find \(u\) in a noisy signal
Algorithmics:
finding quasiperiodic factorizations
Algorithmics: delay embeddings

- Choosing dimension of the ambient space is tricky:
  - Too high or too low dimensionality is a problem
- Consider $u(x) = \cos(x^2)$

A typical delay embedding with too low dimension gets “tangled” with self-intersections...
Algorithmics: delay embeddings

- Choosing dimension of the ambient space is tricky:
  - Too high or too low dimensionality is a problem
- Consider $u(x) = \cos(x^2)$

... too high dimension yields an image diffeomorphic to $\mathbb{R}$: a trivial factorization
Topological estimation

- Generalize to group actions on manifolds!

\[ F(x) = (u(x), u(g_1 x), u(g_2 x), \ldots, u(g_{N-1} x)) \]

Image under generalized delay map: an **immersed** submanifold
How does it work?

- Picking the rotations can be done, but not arbitrarily!
- **Lemma**: If $u : M \to N$ is a smooth map and
  - $M$ is a compact manifold
  - $G$ is a group of diffeomorphisms acting on $M$ transitively

  Then there is a finite set $\{g_1, \ldots, g_m\}$ such that

  $$F(x) = (u(x), u(g_1x), \ldots, u(g_mx))$$

  has constant rank and

  $$\text{rank } dF(x) = \max \text{ rank } du(y) \text{ over all } y \text{ in } M.$$  

- **Theorem**: (R.) Use these in your generalized delay map to obtain a **universal** quasiperiodic factorization
Filtering using Quasiperiodic Factorizations

The QuasiPeriodic Low Pass Filter (QPLPF)
Circumventing bandwidth limits

- Traditional: averaging in a connected window
  - Noise cancellation (Good)
  - Distortion to the signal (Bad)

- QPLPF: **Safely do more** averaging across the **entire** signal using a quasiperiodic factorization first
QPLPF block diagram

Stage 1: Topological estimation
- Quasiperiodic factorization
- Neighbors
- Average along rows

Stage 2: Topological filtering
- Quotient construction
- Averaging filter

Input signal → Quasiperiodic factorization → Quotient construction → Averaging filter → Output signal

Time
Neighbors
Average along rows
QPLPF results

Some low frequency distortion

Extremely stable output amplitude
Compare: standard adaptive filter

Signals at ports of variable-bandwidth LPF filter

Unstable amplitude
Filter performance comparison

- QPLPF combines good noise removal with signal envelope stability
Ocean radar image despeckling

After topological filtering:

- Speckle and contrast improved

![Original Image](20140309 original image)

![Filtered Image](20140309 filtered image)

QPLPF
Next steps

• Can we find the necessary “rotation” group elements algorithmically?
  – The proof that they exist is non-constructive!
  – How many are needed practically (probably more than are required theoretically)?

• Implementations complete for dim $M = 2$… generalize!

• Already tested on ocean SAR images.. now apply to others
For more information

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http://www.drmichaelrobinson.net/